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THE PRINCIPLES OF ANALYTICAL GEOMETRY

DESIGNED
FOR THE USE OF STUDENTS
IN THE UNIVERSITY.

BY

H. P. HAMILTON, M.A. F.R.S.E.

FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

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THE PRINCIPLES

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ANALYTICAL GEOMETRY.

INTRODUCTION.

1. MATHEMATICAL Quantity is of two kinds, Number and Extension. The Science which treats of the former is called Arithmetic; that which treats of the latter, is called Geometry.

Two separate branches of Science are comprehended under the general denomination of Arithmetic; *Arithmetic*, properly so called, and *Universal Arithmetic*, or *Algebra*. Number is denoted by particular characters in the one, and by general characters in the other; but the operations in both are founded upon the same principles.

2. Algebra and Geometry, although conversant with objects essentially different, may yet be rendered subservient to each other in the investigation of mathematical truths. To understand how this is to be effected, we must attend to the following considerations.

In Algebra, Quantity, and the Relations of Quantity, are both denoted by symbols; in Geometry, Quantity is represented by magnitudes resembling those which are the objects of reasoning, as a line by a line, a circle by a circle, and so on; and the various relations of Quantity are expressed in common language.

If the same symbolical notation, therefore, be employed to represent both kinds of Quantity, geometrical questions may then be treated algebraically, and the primary operations of Algebra may be performed geometrically. In the former case,

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therefore the geometrical value of x is a mean proportional between

$$\lambda \text{ and } a+b+c+\ldots-(d+e+\ldots).$$

Take a straight line

$$AC=a+b+\ldots-(d+e+\ldots),$$

produce AC to B, so that CB may $=\lambda$; on AB describe a semi-circle, and from C draw CP at right angles to AB, meeting the circumference in P; then CP is the line required.

$$(2) \quad \text{Let } x=\sqrt{ab};$$

$$\therefore\ x^2=ab,$$

and (x) is a mean proportional between a and b, which may, therefore, be determined by the first case.

$$(3) \quad \text{Let } x=\sqrt{\{ab+cd\}};$$

$$\therefore\ x^2=ab+cd$$

$$=a\left(b+\frac{cd}{a}\right);$$

therefore x is a mean proportional between (a) and the line $b+\frac{cd}{a}$.

$$(4) \quad \text{Let } x=\sqrt{(a^2+b^2)},$$

then x is evidently the hypotenuse of a right-angled triangle whose sides are a and b.

$$\text{If } x=\sqrt{\{a^2+b^2+c^2\}}.$$

$$\text{Find } m^2=b^2+c^2,$$

$$\therefore\ x=\sqrt{(a^2+m^2)},$$

and therefore may be constructed as before.

$$(5) \quad \text{Let } x=\sqrt{(a^2-b^2)} \text{ or } =\sqrt{(a+b)(a-b)}.$$

Here x is a mean proportional between $a+b$ and $a-b$, and may therefore be determined.

Now as the sign of the second term depends upon that of x, it will be sufficient to construct the roots of

$$x^2 - 2px \pm m^2,$$

Since these roots, when taken negatively, will represent the roots of $x^2 + 2px \pm m^2$.

The equation being resolved, $x = p \pm \sqrt{p^2 \mp m^2}$.

(1). To construct $x = p \pm \sqrt{p^2 - m^2}$.

Assume (fig. 2.) any straight line $AB = p$, and upon it describe a semi-circle; insert the line $AC = m$, join BC, and from B as a centre, with a radius $= BC$, describe a circle cutting BA in P, and AB produced in p. Then the roots required will be AP and Ap.

For $AP = AB - BP = AB - \sqrt{AB^2 - AC^2} = p - \sqrt{p^2 - m^2}$,

and $Ap = AB + Bp = AB + \sqrt{AB^2 - AC^2} = p + \sqrt{p^2 - m^2}$.

When $p > m$, the two roots are real and positive, and may therefore be constructed, since it is always possible to place in a semi-circle a line m, which is less than the diameter p.

When $p = m$, the roots are each $= p$, a circumstance which is indicated by the construction; for the line AC being now equal to the diameter AB, the point C falls on B; therefore $BC = 0$, and the points P and p coincide with B; therefore AP and Ap each equal AB.

When $p < m$, the roots are imaginary, which may also be inferred from the failure of the construction in this case, since a line greater than the diameter cannot be inserted in a circle.

Lastly, when $m = 0$, the two roots are $p - p$ or 0, and $p + p$ or $2p$; and in this case the points C and P coinciding with A,

$$AP = 0, \text{ and } Ap = 2p.$$

(2) To construct $x = p \pm \sqrt{p^2 + m^2}$.

Assume, as before, (fig. 3.) $AB = p$, from A draw AC at right angles to AB, and $= m$; join BC, and from B as a centre, with radius BC, describe a circle cutting AB, produced both ways, in P and p. Then AP and Ap will represent the roots.

For $Ap = Bp + BA = \sqrt{BA^2 + AC^2} + BA = \sqrt{p^2 + m^2} + p$,
and $AP = BP - BA = \sqrt{BA^2 + AC^2} - BA = \sqrt{p^2 + m^2} - p$.

It is here sufficient to assign the *absolute* value of the second root AP, since the direction in which it is measured from A, determines it to be negative relatively to AP; and therefore AP will actually represent

$$-(\sqrt{p^2 + m^2} - p), \text{ or } p - \sqrt{p^2 + m^2}.$$

11. A geometrical problem is said to be *determinate* or *indeterminate*, according as it admits of a finite or of an infinite number of solutions.

For example, if it be required to divide a given straight line into two such parts, that the rectangle contained by the whole line and one of the parts, shall be equal to the square of the other part; this problem is determinate.

If on the other hand, it were proposed to find a point such, that the lines drawn to it from the extremities of a given line, should form a right angle, it is evident that there will be an indefinite number of such points, and that they will lie in the circumference of a semi-circle described upon the given line. This problem, therefore, is indeterminate.

In like manner, the equation which is derived from translating the conditions of a problem into the language of Algebra, will be determinate or indeterminate, according to the nature of the proposed question.

12. Of the two kinds of problems above specified, the first relates solely to the *magnitude* and *position* of geometrical quantities, whilst the second extends to the consideration of

their *figure*. It is problems of the latter class, those, namely, that lead to indeterminate equations, which form the most important and extensive department of Analytical Geometry.

13. The following Treatise is divided into two Parts, corresponding to plane and to solid Geometry.

Part I, in which the objects of investigation are supposed to lie in the same plane, is entitled Analytical Geometry of Two Dimensions.

Part II, in which the objects treated of are situated in different planes, is entitled Analytical Geometry of Three Dimensions.

◆

PART I.

ANALYTICAL GEOMETRY OF TWO DIMENSIONS.

SECTION I.

ON DETERMINATE PROBLEMS.

CHAP. I.

Preliminary Notions.

14. When a geometrical question is proposed for solution, we conceive it to be already resolved, and denote the known and the unknown lines by corresponding letters. We next consider attentively the nature and conditions of the question, and thence derive the means of expressing in two different ways, one or other of the quantities concerned. These two identical expressions being put equal to one another, we obtain an equation, the root or roots of which, when found by the ordinary rules of Algebra, and then constructed geometrically, furnish the required solution.

15. Particular attention must be paid to the *selection of the unknown quantity*, as it is upon this, that the simplicity and elegance of the solution mainly depend. The following rule will serve, in general, to guide us in our choice. "When two or more of the quantities sought produce equations exactly alike, then we

must reject both, and substitute in their place some other quantity, similarly related to each, as their sum or difference, or the mean proportional between them." The utility of this rule will be frequently exemplified in the two following chapters.

16. The process above described, for resolving any geometrical problem algebraically, is divisible into two parts, of which the first consists in *stating the question*; that is, in reducing it to an equation; the second, in exhibiting the root or roots of that equation, by geometrical construction.

On these two parts we shall offer a few remarks.

I. It seldom happens that the reducing equation can be obtained by merely expressing the conditions of the problem in algebraical language.

We must generally endeavour to discover such relations among the lines proposed, as shall fit them most readily for computation. For this purpose, a familiar acquaintance with the Elements of Euclid is requisite. The following propositions, in particular, are frequently serviceable in stating the question:

(1) On the right-angled triangle, I. 47.

(2) On the circle, III. 31, 35, 36.

(3) On similar triangles, VI. 4, 5, 6, 7, 8.

II. From the foregoing investigations it appears, that the simple and quadratic equations, may always be constructed geometrically; the former by the intersection of straight lines, the latter by the intersection of the straight line and circle. The converse of this, as will afterwards be shewn, holds true; namely, that the straight line and the circle in Geometry, are represented by the simple and the quadratic equation, respectively, in Algebra.

Now such problems alone are accounted purely geometrical, as can be resolved by means of the *rule* and *compass*, or, in other words, by means of the straight line and circle. Consequently, every problem of elementary Geometry will lead, when expressed algebraically, to a simple or a quadratic equation, the root or roots of which may, therefore, be constructed by the methods above explained.

We shall begin by resolving such problems, as lead to *simple* equations.

CHAP. II.

ON PROBLEMS WHOSE CONSTRUCTION DEPENDS UPON THAT OF A SIMPLE EQUATION.

17. **Prob. 1.** It is required to place a straight line of given length between the sides of a right-angled triangle, so that it may be bisected by the hypotenuse.

Let ABC (fig. 4.) be the triangle, MN the required line which is terminated by the sides in M and N, and bisected by the hypotenuse in P.

Draw BP, and assuming it for the unknown quantity, let it equal x; also let $MN = m$.

Then, since ABC is a right angle, it lies in a semi-circle, of which MN is the diameter. And as $PM = PN$ by hypothesis, P is the centre of the semi-circle, and therefore $PB = PM$ or PN; that is, $x = \frac{1}{2}m$. Whence is derived the following construction.

From B as a centre, with radius $= \frac{1}{2}m$, describe a circle cutting the hypotenuse in P, and make PM, or $PN = \frac{1}{2}m$: then MNP is the line required.

If it were proposed to draw a straight line, terminated by both sides *produced* and bisected by the hypotenuse, we should find, as above, that $BP' = \frac{1}{2}m$, that is, P' will be the second point of intersection with the hypotenuse, of the circle described from B as a centre, with radius $= \frac{1}{2}m$.

The points P, P', therefore, will resolve the problem, when stated generally, as follows, " To place between the sides, or the sides produced, of a right-angled triangle, a straight line of given length, which shall be bisected by the hypotenuse.

18. *Observation.* As this question has *two* answers, it ought

when expressed algebraically, to have led to a quadratic equation; and such would have been the case, if we had selected either AP or BM for the unknown quantity. By assuming however for that quantity the line BP, which has the same value in both cases of the problem, we are enabled to reduce the equation to a simple one. Art. 15.

19. Prob. 2. To divide the base of a triangle into two segments, proportional to the adjacent sides.

Let ABC (fig. 5.) be the triangle, AD, DB, the required segments.

Assume $BD=x$, and let $BC=a$, $CA=b$, $AB=c$.

Then by hypothesis, $\frac{AD}{DB}=\frac{AC}{CB}$, or by substitution,

$$\frac{c-x}{x}=\frac{b}{a};$$

$$\therefore ac-ax=bx;$$

$$\therefore x=\frac{ac}{a+b}.$$

Hence (x) is a fourth proportional to a, c, and $a+b$; and may thus be found.

Make $CE=CB=a$, join B, E and draw CD parallel to BE; then BD is the value of x.

For $\frac{BD}{BA}=\frac{EC}{EA}=\frac{a}{a+b}$; $\therefore BD=\frac{ac}{a+b}=\therefore x.$

A similar investigation is applicable when the point is supposed to be in the base produced.

Let D' be the point in this case, and assume $BD'=x'$; then

$$\frac{AD'}{BD'}=\frac{AC}{CB},$$

$$\text{or } \frac{c+x'}{x'} = \frac{b}{a};$$

$$\therefore x' = \frac{ac}{b-a};$$

which may be thus constructed.

Make $CE' = CB = a$, join B, E', and draw CD' parallel to BE', then BD' may be shewn, as before, to be the value of x'.

It is evident that the vertical angle of the triangle is bisected by CD, and the supplement of that angle by CD': hence, conversely, "the two straight lines which bisect the vertical angle and its supplement in any triangle, divide the base and the base produced into segments, which are proportional to the adjacent sides of the triangle."

20. Cor. The line AD' is divided harmonically in the points D and B.

$$\text{For } \frac{AD'}{D'B} = \frac{AD}{DB}, \text{ since each } = \frac{AC}{CB};$$

$$\therefore AD' : AD :: D'B : DB$$

$$:: AD' - AB : AB - AD;$$

that is, "the first is to the third as the difference between the first and second is to that between the second and third," which is the property of harmonic progression.

The base AB is therefore an harmonic mean between AD and AD'.

21. Prob. 3. In either side of a given triangle, to find a point, whence if a straight line be drawn parallel to the base, and terminated by the other side, it may be of given length.

Let ABC, fig. 6., be the triangle, and supposing the problem resolved, let M be the required point, and MN the line which is to be equal to a given line (m).

Assume AM for the unknown quantity, and let it $=x$, and also let $AC=b$, $AB=c$.

Then by similar triangles, $\dfrac{AB}{AC}=\dfrac{MN}{MC}$,

or substituting for these ratios their numerical values

$$\frac{c}{b}=\frac{m}{b-x};$$

$$\therefore\ x=\frac{b}{c}(c-m).$$

Whence it appears that (x) is a fourth proportional to b, $c-m$, and c. The point M may therefore be determined by the following construction:

In BA take $BQ=m$, through Q draw QM parallel to BC, meeting AC in M; then M is the point required.

$$\text{For }\ \frac{AM}{AQ}=\frac{AC}{AB}=\frac{b}{c};$$

$$\therefore\ AM=\frac{b}{c}.AQ=\frac{b}{c}(c-m),$$

as it ought to be.

As long as $c>m$, the value of (x) remains positive, and may be assigned by the construction just given.

If $m=c$, then the point M coincides with A, and $x=0$.

If $m>c$, then the value of (x) becomes negative.

$$\therefore\ x=-\frac{b}{c}.(m-c).$$

In order to interpret this expression, we shall resume the equation

$$\frac{c}{b}=\frac{m}{b-x},$$

and since x is now negative, the equation becomes

$$\frac{c}{b} = \frac{m}{b+x}; \therefore b+x = \frac{b}{c}m,$$

but $b+x$ represents here the sum of two lines; which it could not do unless the point required be situated in CA *produced*, that is, in a direction contrary to that in which it was first measured.

To construct the resulting value of x, that is, $-\frac{b}{c}(m-c)$.

In BA *produced* take $BQ' = m$, through Q' draw $Q'M'$ parallel to CB, meeting CA produced in M'; then, if $M'N'$ be drawn parallel to AB, it will be the line required.

From what has been said it appears, that the problem, as stated geometrically, is limited to the particular case in which $m < c$; and that the algebraic solution enables us to generalise the enunciation as follows:—"In either of two lines containing a given angle, to find a point such, that a straight line drawn through it in a given direction, and terminated by the other line, may be of given magnitude."

22. The negative value of (x) is introduced into the above solution by our having arbitrarily assumed the point M to lie *above*, rather than *below* the point A, which implied that m should be less than c. The difference of sign, therefore, in the value of x merely indicates, that when our hypothesis is shifted by (m) becoming $>c$, there will be a corresponding change in the position of M, which must now be measured in the contrary direction.

If we had selected CM for the unknown quantity, its value would always remain positive; since it is measured in the same direction from C whatever be the magnitude of m.

For example, if $CM' = x'$, then

$$\frac{CM'}{M'N} = \frac{CA}{AB} = \frac{b}{c}; \therefore x' = \frac{b}{c}m;$$

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unknown quantity, a line which has not both a *positive* and a *negative* value. If, however, DP be taken equal x, then it is evident that for each point P, P' to the right, there will be a corresponding point to the left, of D. The equation will, in this case, be $x^4 + \frac{2b^2 - a^2}{2} x^2 = b^2 m^2 - \frac{a^4}{16}$, which has *four* roots, two of which are equal in magnitude, but affected with different signs.

28. Prob. 5. Two points A, B, (fig. 15.) being given on a line of indefinite length, to find a third point P, such that $AB \cdot PB$ shall $= AP^2$.

Let the distance of the required point from $A = x$, and $AB = a$.

Then by hypothesis, $AB \cdot PB = AP^2$,

$$\text{or } a(a-x) = x^2 \ldots\ldots\ldots (1);$$

$$\therefore x^2 + ax = a^2;$$

$$\therefore x = -\frac{a}{2} \pm \sqrt{\left(a^2 + \frac{a^2}{4}\right)}.$$

Of these roots the one is positive and $< a$; the other is negative, and in absolute magnitude $> a$: the former therefore resolves the problem, when P is supposed to lie between A and B.

We are now to consider the meaning of the *negative* root.

On referring to equation (1), it will be observed that we have assumed $BP = a - x$, thus restricting the problem to the case in which P falls between A and B. But in order to express the conditions of the question generally, we shall now suppose the point sought to lie *without* AB, either to the left of A, or to the right of B.

1st. Let the point P' be situated to the left of A: then $BP' = a + x$; making this substitution in (1), and resolving the equation, we have

$$x = \frac{a}{2} \pm \sqrt{\left(a^2 + \frac{a^2}{4}\right)}.$$

Here the positive root $= AP'$, and is equal in magnitude to the negative root already found. The two roots therefore of the original equation determine two points P, P', *situated on opposite sides of* A, which resolve the problem.

2nd. Next, suppose that the point required is to the right of A as at P'', then $BP''=x-a$; therefore, substituting in (1), we have $a(x-a)=x^2$,

$$\text{whence } x=\frac{a}{2}\pm\frac{a}{2}\sqrt{-3},$$

which is imaginary; therefore, no point situated to the right of B, can satisfy the conditions. This indeed is evident from merely inspecting the equation $AB.BP''=AP''^2$, for since AB and BP'' are each less than AP'', it is plain that $AB.BP''$ cannot $=AP''^2$.

From this example we perceive, that the impossibility of a problem is not necessarily to be inferred, from the value of the unknown quantity being imaginary. In the present instance, the imaginary solution, merely implies that the conditions of the question are irreconcileable with the supposition that the point P'' is situated to the *right* of B.

Observation. If we assume for the unknown quantity, the distance of the required point from some point M, taken at pleasure, to the left of A and beyond P', we shall then obtain an equation of which both roots will be *positive*.

Thus, let MP the distance sought $=x$, and $MA=b$; then

$$AP=x-b, \text{ and } BP=a+b-x;$$

therefore, substituting these values in the equation

$$AB.BP=AP^2,$$

we have

$$a(a+b-x)=(x-b)^2;$$

$$\therefore\ x^2-(2b-a)x=a^2+ab-b^2;$$

$$\therefore\ x=b-\frac{a}{2}\pm\sqrt{\left(a^2+\frac{a^2}{4}\right)}.$$

Now the point *M* being beyond *P′*, $AM>AP'$; that is,

$$b>\frac{a}{2}+\sqrt{\left(a^2+\frac{a^2}{4}\right)};$$

$$\therefore\ b-\frac{a}{2}>\sqrt{\left(a^2+\frac{a^2}{4}\right)};$$

therefore the two values of (x) above deduced are necessarily positive, and therefore *MP*, *MP′*, are measured in the same direction from the point *M*.

29. **Prob. 6.** Through a given point *A*, (fig. 16.) equidistant from two lines *CX*, *CY*, of known inclination, to draw a straight line, such that the part *PQ*, intercepted by these lines, may be of given length (m).

Supposing the problem resolved, draw *PM*, making the angle $APM=ACQ$, and meeting *AC* in the point *M*; and draw *AB* parallel to *CX*.

Let $AC=a$, $AB=b$, and assume for the unknown quantity x, the distance of *M* from the middle point *O* of *AC*; therefore *AM* will equal $x+\frac{a}{2}$, and $CM=x-\frac{a}{2}$.

Then the triangle *AMP* being similar to each of the triangles *ACQ*, *MCP*, we have

$$\frac{AM}{MP}=\frac{MP}{MC}\ \text{or}\ AM\,.\,MC=MP^2\ \ldots\ldots\ldots\ldots(1).$$

$$\text{also}\ \frac{MP}{PA}=\frac{CQ}{CA}\ \text{or}\ MP=\frac{PA\,.\,CQ}{CA}=\frac{BC\,.\,PQ}{CA}\ \ldots\ldots(2).$$

this value of *MP* being substituted in (1),

$$AM\,.\,MC=\frac{BC^2}{CA^2}\,.\,PQ^2,$$

$$\text{that is, } \left(x + \frac{a}{2}\right)\left(x - \frac{a}{2}\right) = \frac{b^2}{a^2}m^2;$$

$$\therefore x = \pm\sqrt{\left(\frac{a^2}{4} + \frac{b^2}{a^2}m^2\right)}.$$

Take, therefore, OM equal to the positive value of x, from M as a centre, with radius $MP = \frac{b}{a}m$ describe a circle cutting CX, CY in the points P, q, then drawing AP, Aq, the intercepted portions PQ, pq, will be each equal to the line required.

The negative value of x results from our having assumed the point M to the *left*, rather than to the *right* of O: so that $MC = x - \frac{a}{2}$; but if M be supposed, agreeably to the general conditions of the question, to lie to the right of O, then MC will equal $x + \frac{a}{2}$: hence it is evident, as in the examples already considered, that OM' (fig. 17.) will represent the negative value of x.

From M', therefore, as a centre, with radius equal $\frac{b}{a}m$, describe a circle cutting CX, CY in the points P', q', then drawing AP', Aq', and producing them to meet AY, AX in Q', p', the lines $Q'P'$, $q'p'$ will each be equal to m.

Cor. When CX, CY are at right angles to one another, then $a^2 = 2b^2$;

$$\therefore x = \pm\sqrt{\tfrac{1}{2}(b^2 + m^2)}.$$

(*Universal Arithmetic*, p. 125, Prob. 24.)

Observation. The above problem admits, nominally, of *four* solutions; but as two points P, p, are determined by the positive, and the other two P', p' by the negative value of x, the number of independent solutions is reduced to two.

We may adduce this as a favourable example of the advantage derived from selecting, for the unknown quantity, a line that shall undergo the fewest possible changes in the various positions of the figure.

If either CP or Cq had been represented by x, then x would have been positive for P or q, and negative for p or Q: therefore, in both cases, the resulting equation would have been of the fourth degree. But if OM or AM be assumed for the unknown quantity, then each remains the same in magnitude as well as position for both points P, p: the resulting equation will thus be reduced to the second degree.

It may be added, that OM was chosen in preference to AM, because the former leads to a *pure*, and the latter to an *adfected* quadratic.

30. PROB. 7. Given the chords of two arcs, to find the chord of their sum or difference.

(1) To find the chord of their sum.

Let AC, CB, (fig. 18.) be the given chords of the two corresponding arcs, and AB the chord of their sum. Assume $BC=a$, $CA=b$, $AB=c$, and the radius of the circle $=r$.

Through the point C, draw the diameter CP, and join PA, PB.

Then, since $CAPB$ is a quadrilateral figure inscribed in a circle, we have

$$AB\,.\,CP=AC\,.\,BP+AP\,.\,BC,$$

which equation becomes, by substitution,

$$c\,.\,2r=b\,.\,BP+a\,.\,AP,$$

but, CP being the diameter,

$$BP=\sqrt{CP^2-BC^2}=\sqrt{(4r^2-a^2)},$$

and $AP=\sqrt{(4r^2-b^2)}$; therefore substituting, and dividing by $2r$, we have

$$c=\frac{b}{2r}\sqrt{(4r^2-a^2)}+\frac{a}{2r}\sqrt{(4r^2-b^2)}\ldots\ldots\ldots\ldots(1).$$

(2) To find the chord of their difference.

In this case, we have given (b) and (c), (fig. 19.) to find (a).

Through the point A, therefore, draw the diameter AP; Then, as before,

$$AB \,.\, CP = AC \,.\, PB + AP \,.\, CB,$$

$$\text{or, } c\ \sqrt{(4r^2 - b^2)} = b\ \sqrt{(4r^2 - c^2)} + 2r \,.\, a;$$

$$\therefore a = \frac{c}{2r}\ \sqrt{(4r^2 - b^2)} - \frac{b}{2r}\ \sqrt{(4r^2 - c^2)} \ldots\ldots\ldots (2).$$

Scholium. The principles which serve to determine the signification of the negative sign, may be collected from a review of the foregoing Propositions.

In the statement of a geometrical question, the unknown letter (x) is always understood to represent an *absolute quantity*. When the solution therefore assigns a *negative* value to x, we infer, either that the negative root has no geometrical signification, or that it results from our having only partially expressed the conditions of the question.

(1) The nature of the problem will enable us to ascertain in what cases the negative root is to be considered as having no geometrical signification. Thus, in a plane triangle, if p be the perpendicular dropped from any angle C, upon the opposite side c, then

$$p = \pm \frac{1}{2c}\ \sqrt{\{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4)\}}.$$

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According to this formula, the perpendicular has *two* values, the one positive, the other negative, which is absurd. We retain, therefore, only the former, and reject the latter, as being altogether symbolical.

(2) The negative root, when it is significant, may be made to disappear, and a positive value of the unknown quantity obtained, by modifying, in a suitable manner, the statement of the question.

In all cases, however, where the unknown quantity represents *the distance between two points, one of which is fixt, estimated* *along a line given in position,* the process just mentioned is unnecessary, and the negative root may be directly interpreted by the following rule: "if the positive values of the unknown quantity be measured in either direction from the fixt point, the negative values are to be measured in the *opposite* direction from the same point."

CHAP. IV.

ON THE APPLICATION OF TRIGONOMETRY TO THE SOLUTION OF GEOMETRICAL PROBLEMS.

We shall begin by establishing the fundamental formula of Plane Trigonometry.

31. **Prob. 1.** Let A and B be any two angles, to prove that

$$\sin(A+B)=\sin A\cos B+\cos A\sin B.$$

Let the angles at the base of the triangle ABC represent the given angles; and let $BC=a$, $CA=b$, and $AB=c$, then dropping the perpendicular CD, we have

$$AB=AD+DB=AC\cos A+BC\cos B,$$

$$\text{or } c=b\cos A+a\cos B;$$

$$\therefore\ 1=\frac{b}{c}\cos A+\frac{a}{c}\cos B$$

$$=\frac{\sin B}{\sin C}.\cos A+\frac{\sin A}{\sin C}.\cos B,$$

because the sines of the angles are as the opposite sides.

But since $A+B+C=\pi$, $\sin C=\sin\{\pi-(A+B)\}=\sin(A+B)$;

$$\therefore\ \sin(A+B)=\sin A\cos B+\cos A\sin B.$$

Hence are immediately deducible the expressions for

$$\sin(A-B)\text{ and }\cos(A\pm B).$$

Note. The formula expressing the value of sin $(A \pm B)$, may be derived in a very simple manner from Prop. 7. Chap. iii.

32. Prob. 2. The three sides of a plane triangle being given, to find the perpendicular dropped from any one of the angular points on the opposite side.

Let A, B, C denote the angles, and a, b, c the opposite sides respectively. Then if (p) be the perpendicular dropped from C upon c, we have

$$\begin{aligned} p^2 &= AC^2 \sin^2 A \\ &= b^2 (1 - \cos^2 A) \\ &= b^2 \left\{1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2\right\} \\ &= \frac{1}{4c^2} \{4b^2c^2 - (b^2 + c^2 - a^2)^2\}; \end{aligned}$$

hence, expanding the trinomial within the bracket, reducing the expression, and extracting the square root, we have

$$p = \frac{1}{2c} \sqrt{\{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4)\}}.$$

Cor. 1. The sine of any one of the angles, and the area of the triangle, may therefore be expressed as a symmetrical function of the sides.

Cor. 2. The area of the triangle

$$= \tfrac{1}{4} \sqrt{\{4b^2c^2 - (b^2 + c^2 - a^2)^2\}}.$$

Now, the quantity under the radical being the difference of two squares

$$\begin{aligned} &= \{2bc + b^2 + c^2 - a^2\} \{2bc - (b^2 + c^2) + a^2\} \\ &= \{(b + c^2) - a^2\} \{a^2 - (b - c)^2\} \\ &= (b + c + a)(b + c - a)(a + b - c)(a - b + c); \end{aligned}$$

if therefore, $\frac{a+b+c}{2}$ be assumed $= s$,

$$\text{the area will } = \sqrt{s(s-a)(s-b)(s-c)}.$$

In order that this quantity may be real, $s-a$, $s-b$, $s-c$, must be all of them positive, or one must be positive, and the other two negative. But, $s-a$, $s-b$, cannot both be negative, for in that case $2s$, would $<a+b$, which is impossible. It follows therefore that all three must be positive: hence, $s>a$, $s>b$, and $s>c$, or, substituting for s its value, $b+c>a$, $a+c>b$, and $a+b>c$, which are the essential conditions of a triangle, that *any two of its sides must be together greater than the third.*

PROB. 3. To find a point in the circumference of a circle, such that the lines drawn to it from the extremities of a given chord, may have a given ratio.

Let AB be the given chord, and P the point required, (fig. 20.)

Take $AB=a$, $\frac{PB}{PA}=m$, $AP=x$, and $\angle APB=\alpha$.

$$\text{Then } AB^2=AP^2+PB^2-2AP.PB.\cos P...(1);$$

therefore, by substitution, $a^2=x^2+m^2x^2-2mx^2\cos\alpha$;

$$\therefore x=\pm\frac{a}{\sqrt{\{1-2m\cos\alpha+m^2\}}}.$$

The positive value of x determines the point P; the negative has no meaning whatever.

The question, however, *has two* solutions: but, in order to obtain the second, we must extend the hypothesis from which equation (1) was derived.

In that equation we supposed the point P to be situated *above* AB, but as it may equally be situated below AB, $\cos P'$ will, in this case, $=\cos(\pi-\alpha)=-\cos\alpha$;

$$\therefore AP' \text{ will } =\pm\frac{a}{\sqrt{\{1+2m\cos\alpha+m^2\}}},$$

in which the negative root, as before, is without meaning.

The two solutions, therefore, of the problem, are comprehended in the formula

$$x = \frac{a}{\sqrt{\{1 \pm 2m \cos \alpha + m^2\}}}.$$

Observation. These two solutions may be obtained by means of a simple equation, if we assume for the unknown quantity the angle ABP.

Let therefore $ABP = \phi$, then $PAB = \pi - (\alpha + \phi)$.

$$\text{Now } \frac{PB}{PA}, \text{ or } m = \frac{\sin PAB}{\sin PBA} = \frac{\sin (\alpha + \phi)}{\sin \phi}$$

$$= \frac{\sin \alpha \cos \phi + \cos \alpha \sin \phi}{\sin \phi};$$

$$\therefore m = \sin \alpha \cot \phi + \cos \alpha;$$

$$\therefore \cot \phi = \frac{m - \cos \alpha}{\sin \alpha}.$$

In like manner, if P be supposed to be *below* AB,

$$\cot \phi = \frac{m + \cos \alpha}{\sin \alpha};$$

therefore, both solutions are comprehended in the formula

$$\cot \phi = \frac{m \mp \cos \alpha}{\sin \alpha}.$$

33. PROB. 4. In a plane triangle, given the perpendiculars dropped from the angular points on the opposite sides, to find the angles and the sides.

Let the perpendiculars dropped from A, B, C upon a, b, c, be denoted by α, β, γ, respectively.

Then 2 area of the triangle $= a \times \alpha = b \times \beta = c \times \gamma$;

$$\therefore b = a \frac{\alpha}{\beta}, \quad c = a \frac{\alpha}{\gamma}.$$

Now $\cos A = \frac{b^2+c^2-a^2}{2bc}$; therefore substituting for b, c their values just found,

$$\cos A = \frac{a^2\frac{a^2}{\beta^2}+a^2.\frac{a^2}{\gamma^2}-a^2}{2a^2.\frac{a^2}{\beta\gamma}}$$

$$= \frac{a^2\alpha^2(\beta^2+\gamma^2)-a^2\beta^2\gamma^2}{2a^2\alpha^2\beta\gamma}$$

$$= \frac{\beta}{2\gamma}+\frac{\gamma}{2\beta}-\frac{\beta\gamma}{2\alpha^2}\ldots\ldots(1).$$

In like manner,

$$\cos B = \frac{\alpha}{2\gamma}+\frac{\gamma}{2\alpha}-\frac{\alpha\gamma}{2\beta^2}\ldots\ldots(2),$$

$$\cos C = \frac{\alpha}{2\beta}+\frac{\beta}{2\alpha}-\frac{\alpha\beta}{2\gamma^2}\ldots\ldots(3).$$

The angles are therefore known.

Again,

$$2\text{ area} = ab\sin C = b\times\beta; \quad \therefore\ a = \frac{\beta}{\sin C};$$

In like manner, $b = \frac{\alpha}{\sin C}$, and $c = \frac{\alpha}{\sin B}$, therefore the sides are known.

34. **Prob. 5.** Given the base, the vertical angle, and the sum or the difference of the sides, to find the sides.

1. Let the sum of the sides be given.

Let $AC+BC=2m$, $AC-BC=2x$;

$$\therefore\ AC=m+x,\ BC=m-x.$$

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Now $AB^2 = AC^2 + BC^2 - 2\,AC \,.\, BC \cos C$;

therefore by substitution,

$$\begin{aligned} c^2 &= (m+x)^2 + (m-x)^2 - 2\,(m^2 - x^2)\cos C \\ &= 2\,(m^2 + x^2) - 2\,(m^2 - x^2)\cos C \\ &= 2\,m^2\,(1 - \cos C) + 2\,x^2\,(1 + \cos C) \\ &= 2\,m^2 \,.\, 2\sin^2\frac{C}{2} + 2\,x^2 \,.\, 2\cos^2\frac{C}{2} \\ &= 4\cos^2\frac{C}{2}\left(m^2\tan^2\frac{C}{2} + x^2\right); \end{aligned}$$

$$\therefore\ x^2 = \frac{c^2}{4}\sec^2\frac{C}{2} - m^2\tan^2\frac{C}{2};$$

$$\therefore\ x = \pm\sqrt{\left\{\frac{c^2}{4}\sec^2\frac{C}{2} - m^2\tan^2\frac{C}{2}\right\}}.$$

The sides are therefore

$$m + \sqrt{\left\{\frac{c^2}{4}\sec^2\frac{C}{2} - m^2\tan^2\frac{C}{2}\right\}},$$

$$m - \sqrt{\left\{\frac{c^2}{4}\sec^2\frac{C}{2} - m^2\tan^2\frac{C}{2}\right\}}.$$

The solution is similar when the difference of the sides is given.

The utility of the rule laid down in Art. 15. is here exemplified, the operation being considerably abridged by our having selected for the unknown quantities the *difference of the sides.*

35. **Prob.** 6. Given the base, the vertical angle, and the perpendicular dropped from it upon the base, to find the sides.

Bisect AB in D, and let the angles into which the perpendicular CP divides the vertical angle be denoted by α, β.

Let $DP = x$, $PC = p$,

$$\text{then } \tan C = \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\,.\tan\beta}.$$

$$\text{Now } \tan\alpha = \frac{AP}{PC} = \frac{\frac{c}{2}+x}{p}, \quad \text{and } \tan\beta = \frac{\frac{c}{2}-x}{p};$$

therefore by substitution,

$$\tan C = \frac{\frac{1}{p}\left(\frac{c}{2}+x+\frac{c}{2}-x\right)}{1-\frac{1}{p^2}\left(\frac{c}{2}+x\right)\left(\frac{c}{2}-x\right)}$$

$$= \frac{pc}{p^2-\frac{c^2}{4}+x^2};$$

$$\therefore\ p^2-\frac{c^2}{4}+x^2 = \frac{pc}{\tan C} = pc\,.\cot C;$$

$$\therefore\ x^2 = pc\cot C - p^2 + \frac{c^2}{4};$$

$$\therefore\ x = \pm\sqrt{\left\{pc\cot C - p^2 + \frac{c^2}{4}\right\}};$$

$$\therefore\ AP = \frac{c}{2} \pm \sqrt{\left\{pc\cot C - p^2 + \frac{c^2}{4}\right\}} = c', \text{ suppose};$$

$$\therefore\ AC = \sqrt{(AP^2+PC^2)} = \sqrt{(c'^2+p^2)},$$

and therefore is known; and in like manner, *BC* is known.

A similar method of solution is applicable to the following problem.

Given the vertical angle and the segments of the base made by a perpendicular dropped from that angle; to find the sides.

36. Prob. 7. Given the difference of the angles at the base, the base and the perpendicular, to find the sides.

Let θ be the given difference, and $DP = x$, then $\tan\theta = \tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$;

now $\tan A = \dfrac{CP}{PA} = \dfrac{p}{\frac{c}{2} + x}$, and $\tan B = \dfrac{p}{\frac{c}{2} - x}$;

$$\therefore \tan\theta = \frac{\dfrac{p}{\frac{1}{2}c + x} - \dfrac{p}{\frac{1}{2}c - x}}{1 + \dfrac{p^2}{\frac{c^2}{4} - x^2}} = \frac{-2px}{\frac{c^2}{4} - x^2 + p^2};$$

$$\therefore \tan\theta \left\{p^2 + \frac{c^2}{4} - x^2\right\} = -2px;$$

$$\therefore x^2 - 2px \cot\theta = p^2 + \frac{c^2}{4}.$$

this quadratic being solved,

$$x = p\cot\theta \pm \sqrt{\left\{p^2 \operatorname{cosec}^2\theta + \frac{c^2}{4}\right\}};$$

$$\text{hence } AC = \sqrt{\left\{\left(\frac{c}{2} + x\right)^2 + p^2\right\}};$$

$$\text{and } BC = \sqrt{\left\{\left(\frac{c}{2} - x\right)^2 + p^2\right\}}$$

are known.

37. **Prob. 8.** Given the angles of a triangle, and the perpendiculars dropped on the sides from a given point within or without the triangle, to find the sides.

Let Q be the given point, (fig. 21.) $Q\alpha$, $Q\beta$, $Q\gamma$, the perpendiculars, and let these perpendiculars be denoted respectively by α, β, γ;

then area of the triangle $= AQB + AQC \pm BQC$,

according as Q is within or without the triangle;

$$\therefore \tfrac{1}{2}bc \sin A = \tfrac{1}{2}\{c\gamma + b\beta \pm a\alpha\} \ldots\ldots\ldots (1).$$

Now $\frac{b}{a} = \frac{\sin B}{\sin A}$; $\therefore b = a\frac{\sin B}{\sin A}$, and $c = a\frac{\sin C}{\sin A}$;

therefore by substitution in (1),

$$\frac{a^2 \sin B \sin C}{\sin A} = a\left\{\gamma\frac{\sin C}{\sin A} + \beta\frac{\sin B}{\sin A} \pm \alpha\right\}$$

$$= \frac{a}{\sin A}\{\gamma \sin C + \beta \sin B \pm \alpha \sin A\};$$

$$\therefore a = \frac{1}{\sin B \sin C}\{\gamma \sin C + \beta \sin B \pm \alpha \sin A\}.$$

In like manner

$$b = \frac{1}{\sin A \sin C}\{\gamma \sin C + \alpha \sin A \pm \beta \sin B\},$$

$$c = \frac{1}{\sin A \sin B}\{\alpha \sin A + \beta \sin B \pm \gamma \sin C\}.$$

Whence the sides are determined.

Cor. When the point Q is equidistant from the sides, it becomes the centre of the circle that touches the sides internally or externally. Hence, if (r) be the radius of the inscribed circle,

$$r = \frac{\text{area}}{\tfrac{1}{2}(a+b+c)} = \frac{\text{area}}{s}, \; s \text{ being put} = \tfrac{1}{2}(a+b+c).$$

In like manner, it immediately follows from equation (1), that if r', r'', r''' be the radii of the circles that touch the sides *externally*.

$$r' = \frac{\text{area}}{s-a}, \; r'' = \frac{\text{area}}{s-b}, \; r''' = \frac{\text{area}}{s-c}.$$

Hence, multiplying the four radii together, we have

$$r.r'.r''.r''' = \frac{(\text{area})^4}{s\,(s-a)\,(s-b)\,(s-c)},$$

$$\text{but } s\,(s-a)\,(s-b)\,(s-c) = (\text{area})^2;$$

$$\therefore\ r.r'.r''.r''' = (\text{area})^2.$$

38. **Prob. 9.** Given the three sides of a triangle, to find the radius of the circumscribed circle.

Let O (fig. 22.) be the centre, drop the perpendicular OD on AB, which will, therefore, be bisected in D.

Assume $AO = R$.

$$\text{Then } R = \frac{AD}{\sin AOP} = \frac{c}{2 \sin AOD} = \frac{c}{2 \sin C} \ \ldots\ldots (1).$$

Now $\sin C = \dfrac{2\ \text{area}}{a\,b}$; therefore by substitution in (1).

$$R = \frac{\frac{1}{4}\,a\,b\,c}{\text{area}} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

or substituting for the area its value deduced from Cor. 1. Prop. 2.

$$R = \frac{\pm\,a\,b\,c}{\sqrt{\{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4)\}}} \ \ldots (3),$$

the positive sign alone is to be used, (Schol. Ch. iii.)

Cor. 1. By the corollary to the last Proposition,

2 area $= r\,(a + b + c)$, and from equation (2),

$$2\ \text{area} = \frac{a\,b\,c}{2R};$$

$$\therefore\ R\,r = \frac{a\,b\,c}{2\,(a + b + c)}.$$

Cor. 2. It appears from equation (1) in the above Proposition, that $\dfrac{c}{\sin C} = 2R$. Hence, *in a plane triangle, the* *constant ratio which any side bears to the sine of the opposite angle is that of the diameter of the circumscribing circle to unity.*

39. Prop. 10. To find the distance between the centres of the inscribed and circumscribed circles.

In any triangle ABC, conceive r to be the centre of the inscribed, R that of the circumscribed circle; let $AR = R$, and assume the distance $Rr = \delta$.

Then in the triangle RAr,

$$Rr^2 = AR^2 + Ar^2 - 2AR \,.\, Ar \cos RAr,$$

but $Ar = \dfrac{r}{\sin \dfrac{A}{2}}$, and $\angle RAr = \frac{1}{2}(B \sim C)$;

$$\therefore \delta^2 = R^2 + \frac{r^2}{\sin^2 \dfrac{A}{2}} - \frac{2Rr}{\sin \dfrac{A}{2}} \,.\, \cos \tfrac{1}{2}(B - C),$$

if B be supposed $> C$; therefore multiplying out, transposing, and observing that

$$\sin \frac{A}{2} = \cos \tfrac{1}{2}(B + C),$$

we have $(\delta^2 - R^2) \sin^2 \dfrac{A}{2} = r^2 - 2Rr \cos \frac{1}{2}(B - C) \cos \frac{1}{2}(B + C)$,

or, since $1 - \cos A = 2 \sin^2 \dfrac{A}{2}$,

and $\cos B + \cos C = 2 \cos \frac{1}{2}(B + C) \cos \frac{1}{2}(B - C)$,

$$\tfrac{1}{2}(\delta^2 - R^2)(1 - \cos A) = r^2 - Rr(\cos B + \cos C) \;\ldots\ldots\; (1).$$

In like manner,

$$\tfrac{1}{2}(\delta^2 - R^2)(1 - \cos B) = r^2 - Rr(\cos A + \cos C) \;\ldots\ldots\; (2);$$

therefore subtracting (2) from (1),

$$(\delta^2 - R^2)(\cos B - \cos A) = 2rR(\cos A - \cos B),$$

$$\text{or } \delta^2 = R^2 - 2rR.$$

Whence the distance required is known.

40. **Prob. 11.** Given the sides of a quadrilateral figure inscribed in a circle, to find the angles and the diagonals.

Conceive $ABCD$ to be the given quadrilateral, and let

$AB=a,\ BC=b,\ CD=c,\ DA=d,\ BD=m,\ AC=n.$

(1) To find the angles

$$\text{In the triangle } ABC,\ n^2=a^2+b^2-2ab\cos B,$$
$$\text{and in } ADC,\ n^2=c^2+d^2-2cd\cos D$$
$$=c^2+d^2+2cd\cos B;$$
$$\text{since } \cos D=\cos(\pi-B)=-\cos B;$$

$$\text{therefore eliminating } n^2,\ \cos B=\frac{a^2+b^2-(c^2+d^2)}{2(ab+cd)};$$

$$\text{therefore } 1+\cos B=1+\frac{a^2+b^2-(c^2+d^2)}{2(ab+cd)}$$
$$=\frac{(a+b)^2-(c-d)^2}{2(ab+cd)}$$
$$=\frac{\{a+b+c-d\}\{a+b+d-c\}}{\frac{1}{2}(ab+cd)};$$

$$\text{assume } s=\tfrac{1}{2}(a+b+c+d),$$

$$\text{then } 1+\cos B=\frac{2}{ab+cd}(s-c)(s-d)\ \ldots\ldots (1).$$

$$\text{Similarly, } 1-\cos B=\frac{2}{ab+cd}(s-a)(s-b)\ \ldots\ldots (2);$$

$$\therefore \sin B=\sqrt{(1+\cos B)(1-\cos B)}$$
$$=\frac{2}{ab+cd}\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}.$$

In like manner,

$$\sin A=\frac{2}{ad+bc}\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}},$$

hence, the angles are determined.

(2) Next, to find the diagonals.

The value of $\cos B$ being substituted in the expression for n^2 already deduced, we have

$$n^2 = c^2 + d^2 + 2cd \, . \, \frac{a^2 + b^2 - (c^2 + d^2)}{2(ab + cd)}$$

$$= \frac{1}{(ab + cd)} \left\{ab(c^2 + d^2) + cd(a^2 + b^2)\right\} .$$

$$= \frac{1}{(ab + cd)} (ac + bd)(ad + bc);$$

$\therefore n = \pm \sqrt{\dfrac{(ac + bd)(ad + bc)}{ab + cd}}$. In like manner,

$$m = \pm \sqrt{\frac{(ab + cd)(ac + bd)}{ad + bc}} .$$

The diagonals are, therefore, known.

COR. 1. The area of the quadrilateral $= \frac{1}{2}(ab + cd) \sin B$,

and, therefore, $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$.

COR. 2. From the second part of the Proposition,

$$m \times n = \sqrt{(ac + bd)} \times \sqrt{(ac + bd)} = ac + bd;$$

that is, *the rectangle contained by the diagonals is equal to the sum of the rectangles contained by the opposite sides*; also

$$\frac{m}{n} = \sqrt{\frac{(ac + bd)(ad + bc)}{ab + cd}} \times \sqrt{\frac{ad + bc}{(ab + cd)(ac + bd)}}$$

$$= \frac{ad + bc}{ab + cd};$$

that is, *the two diagonals are to each other as the sums of the rectangles contained by the sides that meet at their extremities*. Legendre III. 33.

COR. 3. If (r) be the radius of the circumscribed circle, then, (38.)

$$r = \frac{n}{2 \sin D} = \sqrt{\frac{(ac + bd)(ad + bc)(ab + cd)}{(s-a)(s-b)(s-c)(s-d)}} .$$

41. THEOREM 1. In every polygon, any one of the sides is equal to the sum of the others, multiplied each by the cosine of the angle which it forms with the first.

Let $ABCDE$, (fig. 23.) represent any polygon. Assume $AE=a$, $AB=b$, $BC=c$, $CD=d$, $DE=f$, and let the angle contained by (a) and (b) be designated the angle a, b, &c.; required to prove that

$$a=b\cos a, b+c\cos a, c+d\cos a, d\ldots.$$

From B, C, D drop the perpendiculars Bm, Cn, Dp on AE,

$$\text{then } AE=Am+mn+np+pE\ldots.(1),$$

but $Am=AB\cos BAE=b\cos a, b$, and so on; therefore by substitution,

$$a=b\cos a, b+c\cos a, c+d\cos a, d+\ldots.$$

COR. Multiplying the above equation by (a), we have

$$a^2=ab\cos a, b+ac\cos a, c+ad\cos a, d\ldots$$

$$\text{similarly, } b^2=ba\cos b, a+bc\cos b, c+bd\cos b, d,$$

$$c^2=ca\cos c, a+cb\cos c, b+cd\cos c, d,$$

$$d^2=da\cos d, a+db\cos d, b+dc\cos d, c, \&c.;$$

$\therefore$ subtracting the last three lines from the first, and transposing, we have

$$a^2=b^2+c^2+d^2+\ldots-2\{bc\cos b, c+bd\cos b, d+\ldots\},$$

that is, "*in any polygon, the square of any one of the sides is equal to the sum of the squares of the other sides, diminished by twice the sum of the rectangles under every two of these sides, and the cosine of the contained angle.*"

Note. When d, &c. $=0$, the polygon becomes a triangle, and we obtain the well known formula

$$a^2=b^2+c^2-2bc\cos b, c.$$

42. THEOREM 2. The area of any polygon is equal to half

the sum of the rectangles under every two of its sides, omitting one, and the sine of the contained angle.

Let us take, for example, the pentagon $ABCDE$.

Produce CB, EA, (fig. 24.) to meet in M, and join C, E.

$$\text{Let } AM=m, \quad BM=n.$$

Then the area of the figure $=MEC+EDC-ABM$.

$$\begin{aligned}\text{Now } MEC-ABM &= \tfrac{1}{2}\sin M\,\{(a+m)(c+n)-mn\}\\ &= \tfrac{1}{2}\sin M\,\{ac+an+mc\};\end{aligned}$$

therefore area $=\frac{1}{2}\sin M\,\{ac+an+mc\}+\frac{1}{2}de\sin d,e \ldots (1)$.

$$\text{But } m=b\,\frac{\sin b,c}{\sin M}, \quad \text{and } n=b\,\frac{\sin a,b}{\sin M};$$

therefore substituting in (1), and observing that $\sin M=\sin a,c$, we have

$$\text{area}=\tfrac{1}{2}\,\{ab\sin a,b+bc\sin b,c+ac\sin a,c+de\sin d,e\}.$$

The same may in like manner be proved true of any polygon. (*Lhuilier*, *Polygonometrie*, Ch. i. §. viii.)

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SECTION II.

ON THE STRAIGHT LINE AND CIRCLE.

CHAP. I.

PRELIMINARY PRINCIPLES.

43. In any indeterminate equation between two variables, the value of either of them can only be determined by assigning an arbitrary value to the other. For example, in the equation

$$y = 5x + 2.$$

Suppose $x = 0$, then $y = 2$,

$x = 1$, $\quad y = 7$,

$x = -1$, $\quad y = -3$,

and so on; the number of solutions being evidently unlimited.

44. The various solutions of any indeterminate equation may be represented *geometrically*, as follows.

Let $f(x, y)$ be any indeterminate equation between x and y, (a) an assumed value of x, and (b) the resulting value of y.

Draw two lines AX, AY, of indefinite length, and known inclination, intersecting each other in A (fig. 25.). In AX take $AM = a$, and in AY, take $AN = b$, through M, N, draw MP, NP parallel to AY, AX, meeting in P; then, the point P corresponds to the analytical solution of the proposed equation.

Since the equation admits an unlimited number of solutions, the points P, furnished by each solution, will also be infinite in number; the assemblage of these points will form, therefore, a certain line, straight or curved, which is called the *locus* of the equation $f(x, y) = 0$.

45. Of the two quantities a and b, which represent AM and MP, the former is called the *abscissa*, the latter, the *ordinate* of the given point P: they are both comprehended under the general name of *the co-ordinates* of that point.

The lines AX, AY, are called the *axes*, and the point A the *origin*, of the co-ordinates; the inclination of the axes being altogether arbitrary, it is frequently convenient to suppose them at right angles to each other; in this case, the axes and the co-ordinates are said to be *rectangular*.

When the point is not given, its co-ordinates are expressed by the unknown letters x and y, of which the former denotes an abscissa measured along AX, the latter an ordinate measured along AY: for the sake of brevity, AX will in future be denominated the axis of x, and AY the axis of y.

If the point be situated on the axis of x, then $y = 0$, if on the axis of y, then $x = 0$, and if it coincide with the origin, then x and y both $= 0$.

46. Since (x) is measured from a fixt point A, along a line AX given in position, if the values of x to the right of A be supposed positive, those to the left of A must be considered negative. For a similar reason, if the values of (y), measured along AY be positive, those in the direction AY' must be reckoned negative. Schol. Chap. 3. Sect. I. The signs with which x and y are affected, will therefore enable us to determine in which of the four angles formed by the intersection of the axes, the point P is situated: thus, (fig. 26.)

$x = +a$, $y = +b$ characterizes the point P,

$x = +a$, $y = -b$ . . . . P',

$x = -a$, $y = +b$ . . . . P'',

$x = -a$, $y = -b$ . . . . P'''.

47. Let us now suppose a curve to be traced upon a plane; then, reversing the process by which the locus of a given equation is described, we may deduce, from some known property of the curve, the relation subsisting between the co-ordinates of any one of its points. The equation which expresses this relation, supposing it to be the same for every assumed point, is called the *equation to the curve.*

48. There is yet another method of determining analytically the position of a point upon a plane.

Let P (fig. 25.) be the point, join A, P; and assume

$$AP = r, \quad \angle PAX = \omega, \text{ and } \angle YAX = \alpha.$$

Then, in the triangle AMP,

$$\frac{AM}{AP} = \frac{\sin APM}{\sin AMP} = \frac{\sin PAY}{\sin YAX} = \frac{\sin(\alpha - \omega)}{\sin \alpha};$$

$$\text{therefore } x = r\frac{\sin(\alpha - \omega)}{\sin \alpha}.$$

$$\text{Similarly, } y = r\frac{\sin \omega}{\sin \alpha}.$$

Consequently, since x and y may be expressed in terms of r and ω, these latter quantities will serve to fix the position of P. When this mode of defining a point is employed, A is called the *pole*, AP the *radius vector,* and r and ω *polar co-ordinates;* and the equation which expresses the relation between these co-ordinates at any point of a curve, is called the *polar equation* to the curve.

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CHAP. II.

ON THE EQUATION TO THE STRAIGHT LINE.

49. **Prob. 1.** To find the locus of the equation of the first degree.

The general form of this equation is $Ay+Bx+C=0$,

$$\text{or } y=-\frac{B}{A}x-\frac{C}{A}, \text{ hence, if } a=-\frac{B}{A},\ b=-\frac{C}{A},$$

the equation becomes $y=ax+b$.

Now let $x=0$, then $y=b$,

$$y=0, \qquad x=-\frac{b}{a}.$$

In AY (fig. 27.) take $AB=b$, and in XA *produced*, $AC=\frac{b}{a}$, join C, B, then the straight line CBZ is the locus required.

For in BZ take any point P, draw PM parallel to AY, and BQ parallel to AX, meeting MP in Q.

$$\text{Let } AM=x,\ MP=y.$$

$$\begin{aligned}\text{Then } y=MP&=MQ+QP\\ &=b+QP\ \ldots\ldots\ (1).\end{aligned}$$

But, the triangles PQB, BAC being similar,

$$\frac{PQ}{QB}=\frac{BA}{AC}=\therefore a;$$

$$\therefore PQ=a\,.\,QB=ax.$$

Hence, substituting in (1) this value of PQ, there results

$$y=ax+b.$$

The same relation may be shewn to subsist between the coordinates of any other assumed point in the line. Wherefore CBZ is the locus required.

Cor. 1. Conversely, the equation to any straight line is $y=ax+b$.

Cor. 2. The constant quantity a, denotes the ratio of the ordinate to the abscissa, and may be expressed in terms of the angles which the straight line forms with the axes.

Thus, let the straight line be represented by ρ, and its inclination to the axes of x and y, by the symbols of ρ, x and ρ, y;

$$\text{then } a = \frac{BA}{AC} = \frac{\sin BCA}{\sin ABC} = \frac{\sin \rho, x}{\sin \rho, y};$$

$$\therefore y = \frac{\sin \rho, x}{\sin \rho, y} x + b.$$

When the axes are rectangular,

$$\sin \rho, y = \cos \rho, x;$$

$$\therefore a = \tan \rho, x.$$

Cor. 3. If $b = 0$, the line passes through the origin, and the equation becomes $y = ax$.

Let AZ, (fig. 28.), be the line, then its equation is

$$y = \frac{\sin \rho, x}{\sin \rho, y} x + b, \quad \text{or} = \frac{\sin \rho, x}{\sin (x, y - \rho, x)} x + b,$$

if the symbol x, y denote the angle contained by the axes.

We may hence trace the various positions assumed by AZ, whilst the angle ρ, x is supposed to increase from 0 to π.

Let $\rho, x = 0$;
then $y = 0$ and AZ coincides with AX.

Let ρ, x have any positive value $< x, y$,
then, $\sin \rho, x$, and $\sin (x, y - \rho, x)$, are both positive, and as the former increases, the latter decreases; therefore the value of y increases, and remains positive.

Let $\rho, x = x, y$,

then the coefficient of x becomes $\frac{\sin \rho, x}{0}$, which is infinitely great; and AZ coincides with AY.

Let $\rho, x > x, y$, but $< \pi$,

then $\sin \rho, x$ is positive, and $\sin (x, y - \rho, x)$ is negative; therefore the straight line lies in the direction AZ'.

Let $\rho, x = \pi$,

then $\sin \rho, x = 0$; $\therefore y = 0$, and the line coincides with AX'.

Cor. 4. Since $\frac{\sin \rho, x}{\sin \rho, y} = \frac{BA}{AC} = \frac{b}{c}$, if AC be taken $= c$, the equation to any straight line is

$$y = \frac{b}{c} x + b,$$

or dividing by b, and transposing,

$$\frac{y}{b} - \frac{x}{c} = 1.$$

50. Prob. 2. To find the equation to a straight line in terms of the perpendicular dropped upon it from the origin, and the angles which it forms with the axes.

Let p be the perpendicular, and suppose the symbols p, x and p, y to denote the angles which p makes with the axes of x and of y.

Then, referring to the last Cor.

$$b = \frac{p}{\cos p, x} \text{ and } c = \frac{p}{\cos (\pi - p, x)} = -\frac{p}{\cos p, x};$$

therefore by substitution, we have

$$x \cos p, x + y \cos p, y = p,$$

which is the equation sought.

Cor. 1. When the line passes through the origin,

$$x \cos p, x + y \cos p, y = 0.$$

Cor. 2. Since $\cos p, x = -\sin \rho, x,$

and $\cos p, y = \quad \sin \rho, y,$

we have $y \sin \rho, y - x \sin \rho, x = p.$

51. **Prob. 3.** To determine the position of the straight lines, defined by the equation $y = ax + b$.

The various combinations of sign, of which the equation is susceptible, may be exhibited by writing the equation thus,

$$y = \pm(ax \pm b).$$

(1) Let $y = +ax + b$.

The corresponding line has been already considered in Prob. 1.

(2) Let $y = +ax - b$.

If $x = 0, \quad y = -b,$

$y = 0, \quad x = +\frac{b}{a}.$

Hence, in YA produced, (fig. 29.), take $AB' = b$, and in AX take $AC' = \frac{b}{a}$, then $C'B'z'$ is the line required.

(3) Let $y = -ax + b$,

then, as before, $AB = b$, $AC' = \frac{b}{a}$. And the line required is $C'BZ'$.

(4) Let $y = -ax - b$,

take $AB' = b$, $AC = \frac{b}{a}$.

And the line in this case is $CB'z$,

Hence, the lines defined by $y = \pm (ax + b)$, are CZ, Cz.

Those defined by $y = \pm (ax - b)$, are $C'Z'$, $C'z'$.

CHAP. III.

ON PROBLEMS RELATING TO THE STRAIGHT LINE.

52. **Prob. 1.** Required the equation to a straight line passing through a given point, whose co-ordinates are x', y'.

Any point, of which the co-ordinates are x, y, being assumed in the line, we have

$$y = ax + b,$$

but since x', y' are also the co-ordinates of a point in the same line, they will satisfy this equation;

$$\therefore y' = ax' + b;$$

whence, subtracting the lower from the upper equation, we have

$$y - y' = a(x - x'),$$

which is the equation required.

Note. For the sake of brevity, we shall always designate the point *whose co-ordinates are* x', y', as the point (x', y'), and the straight line *whose equation is* $y = ax + b$, as the straight line $y = ax + b$.

53. **Prob. 2.** Required the equation to the line which passes through *two* given points (x', y'), and (x'', y'').

Since (x'', y'') is a point in the line passing through (x', y'), we have, by the last proposition,

$$y'' - y' = a(x'' - x'); \quad \therefore a = \frac{y'' - y'}{x'' - x'}.$$

Substituting this value of (a) in the equation just referred to, we have

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'),$$

which is the equation sought.

Since *two* points fix the position of a straight line, the coefficient (a) is no longer indeterminate as in the last Problem, and we have therefore expressed it in terms of the co-ordinates of the two given points.

Cor. 1. The equation may be thus written,

$$y = \frac{y'' - y'}{x'' - x'}x + \frac{y'x'' - x'y''}{x'' - x'},$$

which is of the general form $y = ax + b$.

Cor. 2. If there be three points $(x', y') : (x'', y'') : (x''', y''')$, in the same straight line, the equation to the line passing through the first two points, is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'),$$

and that to the line passing through the first, and third, is

$$y - y' = \frac{y''' - y'}{x''' - x'}(x - x').$$

Hence, in order that the three points may be in the same line, we must have

$$\frac{y'' - y'}{x'' - x'} = \frac{y''' - y'}{x''' - x'},$$

$$\text{or } (y'' - y')(x''' - x') = (y''' - y')(x'' - x').$$

54. Prob. 3. Required the co-ordinates of the point of intersection of two lines,

$$y = ax + b, \text{ and } y = a'x + b'.$$

The point of intersection being common to the two lines, the co-ordinates of that point will be the same in both equations. Hence, subtracting the second equation from the first, we have

$$x(a-a')+b-b'=0;\ \therefore\ x=-\frac{b-b'}{a-a'},$$

$$\text{and in like manner, } y=-\frac{a'b-b'a}{a-a'},$$

which are the co-ordinates sought.

Cor. When the lines are parallel, x and y are infinitely great; therefore the denominator of the above fractions $=0$, or $a=a'$.

Observation. The above method of determining analytically the intersection of two straight lines, is applicable to all lines whatever, situated in the same plane.

Thus, let $f(x,y)=0$, and $F(x,y)=0$ be the equations to any two curves, which are supposed to intersect each other; then since the co-ordinates of the points of intersection must satisfy both equations, they will be obtained by elimination. The principle of *elimination,* thus employed, is of most extensive use in Analytical Geometry.

55. **Prob.** 4. Required the distance between two points $(x',y',)$ and (x,y).

Let P, P' be two points, (fig. 30.) draw PM, $P'M'$ parallel to AY, and PQ parallel to AX, meeting $P'M'$ in Q.

Let $PP'=r$.

$$\text{Then } r^2=PQ^2+P'Q^2-2PQ\,.\,P'Q\cos Q$$

$$=PQ^2+P'Q^2+2PQ\,.\,P'Q\cos x,y$$

since $\cos Q=\cos(\pi-x,y)=-\cos x,y$;

but $P'Q=M'P'-MP=y'-y$, and $PQ=x'-x$;

therefore by substitution,

$$r^2=(x'-x)^2+(y'-y)^2+2(x'-x)(y'-y)\cos x,y;$$

therefore $r = \sqrt{\{(x'-x)^2+(y'-y)^2+2(x'-x)(y'-y)\cos x, y\}}$, the distance required.

Cor. 1. If the point P' coincide with the origin, x' and y' are each $= 0$;

therefore the distance from $A = \sqrt{\{x^2 + y^2 + 2xy \cos x, y\}}$.

Cor. 2. When the axes are rectangular,

$$r = \sqrt{\{(x' - x)^2 + (y' - y)^2\}},$$

and when the line passes through the origin,

$$r = \sqrt{(x^2 + y^2)}.$$

56. Prob. 5. To find the length of the perpendicular dropped from a given point (x', y') upon the line

$$y = ax + b.$$

Let P be the given point, (fig. 31.), BZ the given line, and PQ the perpendicular whose length (p) is required.

From A drop the perpendicular Ap upon the line drawn through P parallel to BZ; and let Ap meet BZ in q.

Then, since $Ap = y' \sin \rho, y - x' \sin \rho, x$, (50)

and $Aq = b \sin \rho, y$,

we have $p = (y' - b) \sin \rho, y - x' \sin \rho, x \ldots\ldots (1)$,

or $= (y' - ax' - b) \sin \rho, y \ldots\ldots\ldots\ldots (2)$,

because $a = \dfrac{\sin \rho, x}{\sin \rho, y}$.

Cor. 1. Let the given line pass through the origin, then $b = 0$, and we have

$$p = y' \sin \rho, y - x' \sin \rho, x, \text{ as in Art. 50.}$$

Cor. 2. Let the axes be rectangular,

then $\sin \rho, y = \cos \rho, x = \dfrac{1}{\sec \rho, x} = \pm \dfrac{1}{\sqrt{(1 + a^2)}}$;

$$\therefore p = \pm \frac{y' - ax' - b}{\sqrt{(1 + a^2)}},$$

in which the positive or the negative sign is to be used according as the point P is situated above or below the line BZ.

Let the given line pass through the origin, then $b = 0$, and

$$p = \pm \frac{y' - ax'}{\sqrt{(1 + a^2)}}.$$

Let the given point be the origin, then x' and $y' = 0$,

$$\text{and } p = \mp \frac{b}{\sqrt{(1 + a^2)}},$$

according as the line is below or above AX.

57. Prob. 6. Required the angle formed by the intersection of two given lines.

Through the origin draw two lines parallel respectively to the given lines, and let their equations be

$$y = ax; \quad y = a'x.$$

Then, if the two lines be denoted by ρ, and ρ',

$$\angle\, \rho, \rho' = \angle\, \{\rho, x - \rho', x\};$$

$$\text{therefore } \tan \rho, \rho' = \tan\, \{\rho, x - \rho', x\}$$

$$= \frac{\tan \rho, x - \tan \rho', x}{1 + \tan \rho, x \,.\, \tan \rho', x} \;\ldots\ldots (1).$$

$$\text{Now } a = \frac{\sin \rho, x}{\sin \rho, y} = \frac{\sin \rho, x}{\sin (x, y - \rho, x)}$$

$$= \frac{\sin \rho, x}{\sin x, y \cos \rho, x - \cos x, y \sin \rho, x},$$

or, dividing numerator and denominator by $\cos \rho, x$,

$$= \frac{\tan \rho, x}{\sin x, y - \cos x, y \tan \rho, x};$$

$$\therefore\ a \left\{\sin x, y - \cos x, y \tan \rho, x\right\} = \tan \rho, x;$$

$$\text{therefore } \tan \rho, x = \frac{a \sin x, y}{1 + a \cos x, y};$$

In like manner, $\tan \rho', x = \dfrac{a' \sin x, y}{1 + a' \cos x, y};$

therefore, by substitution in (1), we have

$$\tan \rho, \rho' = \sin x, y \,.\, \frac{a - a'}{1 + (a + a') \cos x, y + a a'},$$

which is the expression required.

Cor. 1. When the lines are perpendicular to each other,

$$1 + (a + a') \cos x, y + a a' = 0,$$

and when they are parallel, $a - a' = 0$.

Cor. 2. When the axes are rectangular, $\sin x, y = 1$, and $\cos x, y = 0$;

$$\text{therefore } \tan \rho, \rho' = \frac{a - a'}{1 + a a'}.$$

Hence, when the lines are perpendicular, $1 + a a' = 0$,

and when they are parallel, $a - a' = 0$.

58. **Prob. 7.** Required the equation to the line which makes a given angle with the line $y = a x + b$.

Let the equation sought be $y = a' x + b'$, in which a' is to be determined.

If m be taken equal $\tan \rho, \rho'$ in the last Prob. we have, on multiplying out,

$$m + m(a+a')\cos x, y + maa' = \sin x, y\,(a - a');$$

$$\therefore \{\sin x, y + m(a + \cos x, y)\}\, a' = a \sin x, y - m(1 + a\cos x, y),$$

whence

$$a' = \frac{a \sin x, y - m(1 + a\cos x, y)}{\sin x, y + m(a + \cos x, y)}.$$

This value of a', when substituted in the equation

$$y = a'x + b',$$

gives the equation required.

Cor. Hence the equation to a line perpendicular to the line $y = ax + b$, is

$$y = -\frac{1 + a\cos x, y}{a + \cos x, y}\,x + b'.$$

When the axes are rectangular, this equation becomes

$$y = -\frac{1}{a}\,x + b'.$$

59. The solution of problems being in general greatly facilitated by the employment of *rectangular* co-ordinates, we shall subjoin a summary of the results which are deducible from the preceding propositions, when adapted to this particular case.

(1) The equation to a line, whose inclination to AX is $\tan^{-1} a$, is*

$$y = ax + b,$$

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* By the expresssion $\tan^{-1} a$ is meant *the angle whose tangent is* (a).

and when this line passes through the origin, its equation becomes

$$y = ax.$$

(2) The equation to a line which passes

(1) Through *one* given point, is $y - y' = a(x - x')$.

(2) Through *two* given points, is $y - y' = \frac{y'' - y'}{x'' - x'}(x - x')$.

(3) If ρ, ρ' be the angle of intersection of two lines,

$$y = ax + b, \quad y = a'x + b',$$

$$\text{then } \tan \rho, \rho' = \frac{a - a'}{1 + aa'},$$

$$\sin \rho, \rho' = \frac{a - a'}{\sqrt{(1 + a^2)(1 + a'^2)}},$$

$$\cos \rho, \rho' = \pm \frac{1 + aa'}{\sqrt{(1 + a^2)(1 + a'^2)}}.$$

the positive sign being used when the angle is acute, the negative when it is obtuse.

(4) The equation to a straight line drawn through a given point (x', y'), and making an angle $\tan^{-1} m$, with the line $y = ax + b$, is

$$y - y' = \frac{a - m}{1 + am}(x - x').$$

Hence, (1) When the lines are perpendicular,

$$y - y' = -\frac{1}{a}(x - x').$$

(2) When they are parallel,

$$y - y' = a(x - x').$$

(5) If (r) be the distance between two points (x, y) and (x', y'), then

$$r = \sqrt{\{(x' - x)^2 + (y - y')^2\}}.$$

When x' and $y'=0$, $r=\sqrt{(x^2+y^2)}$, which therefore expresses the distance of a point from the origin.

(6) If (p) be the perpendicular dropped from a given point (x', y') on the straight line $y=ax+b$, then

$$p=\pm\frac{y'-ax'-b}{\sqrt{(1+a^2)}}.$$

These formulas the Student will find it convenient to commit to memory.

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CHAP. IV.

APPLICATION OF THE EQUATION TO THE STRAIGHT LINE TO EXAMPLES.

60. Prop. 1. If from the middle points of the sides of a triangle, straight lines be drawn at right angles to the sides, they will meet in the same point.

Let the sides of the triangle *ABC* be bisected in *M, N, P*, (fig. 32.), and from these points draw *Mm, Nn, Pp* at right angles to the sides; it is required to prove that they will have a common point of intersection; or that any two of them *Nn, Pp* will meet the third *Mm* in the same point.

We shall here employ rectangular co-ordinates, and shall suppose, for the sake of simplicity, that they originate at *A*, and that the axis *AX* coincides with *AB*.

Let the co-ordinates of *C* be α, β

then those of *N* and of *P* will be $\frac{\alpha}{2}, \frac{\beta}{2}; \frac{\alpha+c}{2}, \frac{\beta}{2}$, respectively.

Now when *Nn, Pp* meet *Mm* in *any* point, the abscissas alone in the equations to these two lines will have the same value, *AM* or $\frac{c}{2}$; but when they meet *Mm* in the *same* point, the ordinates will likewise be identical.

Our object, therefore, is, first to find the equations to *Nn, Mm*; and next, to substitute in each, $\frac{c}{2}$ for x; then, if the resulting values of y be identical, the truth of the Proposition will be established.

The equation to Nn is (52) of the form

$$y - \frac{\beta}{2} = m\left(x - \frac{a}{2}\right),$$

but Nn being at right angles to AC, whose inclination to AX $= \tan^{-1}\frac{\beta}{a}$,

$$m = -\frac{a}{\beta}, \text{ (Cor. 2. Art. 57.)}$$

therefore the equation to Nn is

$$y - \frac{\beta}{2} = -\frac{a}{\beta}\left(x - \frac{a}{2}\right);$$

but when Nn meets Mm, $x = \frac{c}{2}$; therefore the equation becomes, in this case,

$$y - \frac{\beta}{2} = -\frac{a}{\beta}\cdot\frac{c-a}{2} \dots\dots\dots (1).$$

Again, the equation to Pp is of the form

$$y - \frac{\beta}{2} = m'\left\{x - \frac{a+c}{2}\right\},$$

but $m' = \frac{c-a}{\beta}$, since Pp is at right angles to BC; therefore the above equation becomes

$$y - \frac{\beta}{2} = \frac{c-a}{\beta}\left\{x - \frac{a+c}{2}\right\}.$$

Let Pp meet Mm, then $x = \frac{c}{2}$; therefore the equation is reduced to

$$y - \frac{\beta}{2} = \frac{c-a}{\beta}\cdot -\frac{a}{2} \dots\dots\dots (2).$$

On comparing equations (1) and (2) it appears, that the values of y are identical; wherefore the lines Nn, Pp meet Mm in the same point as was to be proved.

61. PROP. 2. The perpendiculars dropped from the angular points of a triangle on the opposite sides, meet one another in the same point.

Let *AM*, *BN*, *CP* be perpendicular to the sides *BC*, *CA*, *AB*, (fig. 33.)

Then, retaining the same system of co-ordinates as in the last Proposition, the equation to *AM* will be of the form $y = mx$; but *AM* being perpendicular to *BC*, whose inclination to *AX* is $\tan^{-1} - \dfrac{\beta}{c-a}$,

$$y = \frac{c-a}{\beta} x;$$

which, when *AM* meets *CP*, becomes

$$y = \frac{c-a}{\beta} a \ldots\ldots (1).$$

Again the equation to *BN* is of the form $y = m'(x-c)$,

$$\text{but } m' = \frac{a}{\beta};\ \therefore y = -\frac{a}{\beta}(x-c),$$

which, when *BN* meets *CP*, becomes

$$y = -\frac{a}{\beta}(a-c) \ldots\ldots (2).$$

The values of y in (1) and (2) being identical, it follows, that the three perpendiculars intersect in the same point.

62. PROP. 3. The lines drawn from the angles of a triangle to the middle points of the opposite sides, intersect in the same point.

Let *AM*, *BN*, *CP*, (fig. 34.) be drawn to the middle points of the opposite sides, and through *C* draw *CX* parallel to *AB*.

In this and the following Proposition, it will be convenient to employ the *oblique* axes *CX*, *CP*, of which the former is supposed the axis of x, the latter that of y.

Let the co-ordinates of *A* be a, β, then those of *B* will be $-a$, β; whence the co-ordinates of *M* and of *N* will be, respectively,

$$\frac{\alpha}{2}, \frac{\beta}{2}; \quad -\frac{\alpha}{2}, \frac{\beta}{2}.$$

The line AM passes through the points $\left(\alpha, \beta, \text{ and } -\frac{\alpha}{2}, \frac{\beta}{2}\right)$; therefore (53) its equation will be

$$y-\beta = \frac{\beta}{3\alpha}(x-\alpha).$$

Let AM meet CP, then $x=0$; therefore the equation becomes

$$y-\beta = -\frac{\beta}{3} \ldots\ldots (1).$$

In like manner, the equation to BN is $y-\beta = -\frac{\beta}{3\alpha}(x+\alpha)$.

Which, when BN meets CP, becomes $y-\beta = -\frac{\beta}{3} \ldots\ldots (2)$.

The values of y in (1) and (2) being identical, the three lines AM, BN, CP intersect in the same point.

Prop. 4. The lines which bisect the three angles of a triangle meet in the same point.

Let AM, BN, CP (fig. 34.) bisect the angles A, B, C, respectively.

The same axes CX, CP being retained, assume

$$AP=\alpha, \quad PB=\alpha', \quad PC=\beta,$$

Then the equation to AM, will be

$$y-\beta = \frac{\sin \rho, x}{\sin \rho, y}(\alpha - x).$$

$$\text{Now } \frac{\sin \rho, x}{\sin \rho, y} = \frac{\sin \frac{1}{2} A}{\sin \frac{1}{2}(A+C)} = \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} B},$$

$$\text{since } \frac{1}{2}(A+C) = \frac{\pi}{2} - \frac{1}{2} B;$$

therefore the equation to AM is $y-\beta = \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} B}(\alpha - x)$.

Let AM meet CP, in which case $x=0$; $\therefore y-\beta = \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} B} \cdot \alpha \ldots (1)$.

Again, the equation to BN is $y-\beta=\dfrac{\sin\frac{1}{2}B}{\cos\frac{1}{2}A}(\alpha'-x)$,

which, when $x=0$, becomes $y-\beta=\dfrac{\sin\frac{1}{2}B}{\cos\frac{1}{2}A}\alpha'$;

$$\text{but } \alpha'=a\,.\frac{\sin A}{\sin B}=a\,.\frac{2\sin\frac{1}{2}A\cos\frac{1}{2}A}{2\sin\frac{1}{2}B\,.\cos\frac{1}{2}B};$$

$$\therefore y-\beta=\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}B}a\ \ldots\ldots\ldots\ldots\ (2).$$

Which is the same value as in (1); therefore the lines meet in the same point.

64. Prop. 5. To find the area of a triangle, in terms of the co-ordinates of its angular points; the axes OX, OY being oblique.

Let the co-ordinates of A, B, C, respectively be

$$x', y' \,:\, x'', y'' \,:\, x''', y''';$$

then, the equation to AB will be (Art. 53. Cor. 1.)

$$y=\frac{y''-y'}{x''-x'}x+\frac{y'x''-x'y''}{x''-x'}.$$

Therefore, the perpendicular (p) dropped from C upon AB will, (56)

$$=\left\{y'''-\frac{y'x''-x'y''}{x''-x'}\right\}\sin\rho,y-x'''\sin\rho,x\ \ldots\ldots\ (1).$$

Now, conceive a line parallel to OX to be drawn through A, meeting the ordinate of B in P. Then,

$$AP=x''-x',\ \text{and}\ BP=y''-y';\ \text{also,}$$

$$\sin\rho,y=\frac{x''-x'}{c}\sin x,y,\ \text{and}\ \sin\rho,x=\frac{y''-y'}{c}\sin x,y,$$

therefore, by substituting in (1),

$$p=\frac{\sin x,y}{c}\{y'''(x''-x')-y'x''+x'y''-x'''(y''-y')\};$$

hence the area of the triangle $=\frac{1}{2}pc$

$$=\tfrac{1}{2}\sin x,y\ \{y'''x''-x'''y''+y'x'''-x'y'''+y''x'-y'x''\}.$$

Cor. If the axes be rectangular, then $\sin x,y=1$; and therefore,

$$\text{area}\ =\tfrac{1}{2}\{y'''x''-x'''y''+x'''y'-y'''x'+y''x'-x''y'\}.$$

CHAP. V.

ON THE EQUATION TO THE CIRCLE.

61. **Prob. 1.** To find the equation to the circle.

The equation sought will be obtained by expressing analytically that the distance between the centre and any point of the circumference is invariable.

Let a circle, whose radius is (r), be referred to any system of oblique axes; suppose x', y', to be the co-ordinates of the centre, and x, y, those of any point in the circumference.

Then, the distance (r) between the two points (x, y): (x', y'), is thus expressed (55),

$$(x-x')^2+(y-y')^2+2(x-x')(y-y')\cos x, y=r^2,$$

which is the equation required.

Cor. When the origin is placed at the centre, x' and $y'=0$;

$$\therefore\ x^2+y^2+2xy\cos x, y=r^2.$$

62. Let the axes be rectangular, then $\cos x, y=0$; and the equation becomes

$$(x-x')^2+(y-y')^2=r^2\ldots\ldots(1).$$

Let the axis of x or of y pass through the centre, then y' or $x'=0$, and the equation becomes respectively

$$\left.\begin{array}{l}(x-x')^2+y^2=r^2\\ \text{or}\quad x^2+(y-y')=r^2\end{array}\right\}\ldots\ldots(2).$$

Let the origin be upon the circumference, then, since

$$x'^2 + y'^2 = r^2,$$

equation (1) becomes

$$x^2+y^2-2xx'-2yy'=0\ldots.(3);$$

and equation (2) becomes, on the same supposition,

$$\left.\begin{array}{l} x^2+y^2-2rx=0 \\ x^2+y^2-2ry=0 \end{array}\right\}\ldots\ldots(4).$$

Let the origin be at the centre, then, x' and y' are each $=0$, and the equation becomes

$$x^2+y^2=r^2\ldots\ldots(5).$$

Observation. The equations most generally used are the two following;

$$x^2+y^2-2rx=0, \text{ or } y^2=2rx-x^2,$$

$$\text{and } x^2+y^2=r^2.$$

63. Prob. 2. To trace the circle from its equation $x^2+y^2=r^2$.

Let CX, CY, be the rectangular axes to which the circle is to be referred (fig. 35.).

(1) Solving the equation in terms of x, we have

$$y=\pm\sqrt{(r^2-x^2)}.$$

Let $x=0$; $\therefore y=\pm r$. If therefore, CD, CE, be each taken $=r$, the circle will cut the axis of (y), in the points D, E.

Let $x=\pm r$; $\therefore y=0$; therefore, the circle cuts the axis of x at the points A, B.

Let $x \angle \pm r$; then, for each assumed value of x, there are two equal values of y with opposite signs: therefore, the axis of x divides the circle into two equal parts.

Let $x > \pm r$,
then the values of y are imaginary, and no point of the circle is situated beyond A or B.

(2) Next solving the equation in terms of y, we have

$$x = \pm \sqrt{(r^2 - y^2)}.$$

By pursuing the same steps as before, it may be shewn, that the circle is divided into two equal parts by the axis of y, and that no point can be situated beyond D or E.

64. **Prob. 3.** To find the polar equation to the circle

The axes being supposed rectangular, and the origin at the centre, the algebraic equation to the circle will be

$$x^2 + y^2 = r^2 \ldots\ldots\ldots (1).$$

Let any point S, (α, β) be assumed for the pole, (fig. 36.) and through S draw SZ parallel to the axis of x, meeting MP in m, and let the ordinate of S meet CX in s; let $SP = \rho$, and angle $PSZ = \omega$.

$$\text{Then } x = CM = Cs + sM = \alpha + \rho \cos \omega,$$

$$\text{and } y = MP = mM + MP = \beta + \rho \sin \omega;$$

therefore substituting these values of x and y in (1), we have

$$(\alpha + \rho \cos \omega)^2 + (\beta + \rho \sin \omega)^2 = r^2,$$

which on being expanded and reduced, becomes

$$\rho^2 + 2\{\alpha \cos \omega + \beta \sin \omega\} \rho + \alpha^2 + \beta^2 - r^2 = 0,$$

the equation required.

Cor. 1. If the pole be situated without the circle, then the polar equation will be

$$\rho^2 - 2\{\alpha \cos w + \beta \sin w\} \rho + \alpha^2 + \beta^2 - r^2 = 0.$$

Cor. 2. Hence it may be proved, that "if two chords cut one another either within or without a circle, the rectangle under the segments of the one is equivalent to the rectangle contained by the segments of the other." *Euc.* III. 35. and Cor. 36.

Resuming the polar equation,

$$\rho^2 \pm 2\{\alpha \cos \omega + \beta \sin \omega\} \rho + \alpha^2 + \beta^2 - r^2 = 0,$$

Let the pole be the given point either within or without the circle, then the two roots of the above equation will be the lines intercepted between the pole and the circumference. Now by the theory of equations, the product of these two lines $= \alpha^2 + \beta^2 - r^2$, and as this quantity is independent of the angle (ω), it will remain the same for all lines drawn through the assumed point. Whence the truth of the proposition is evident.

CHAP. VI.

ON PROBLEMS RELATING TO THE CIRCLE.

65. **Prob. 1.** The circle being referred to oblique axes originating at the centre, to find the equation to the tangent applied at a given point (x', y') of the circumference.

The equation required will be of the form

$$y - y' = A(x - x') \ldots\ldots (1),$$

in which the coefficient A is to be determined.

If a line be drawn from the given point to the centre, its equation will be $y = \frac{y'}{x'}x$; but the tangent is at right angles to this line, *Euc.* III. 18; therefore we have (58),

$$A = -\frac{1 + \frac{y'}{x'}\cos x, y}{\frac{y'}{x'} + \cos x, y} = -\frac{x' + y' \cos x, y}{y' + x' \cos x, y};$$

therefore by substitution in (1),

$$y - y' = -\frac{x' + y' \cos x, y}{y' + x' \cos x, y}(x - x').$$

which is the equation sought.

Cor. 1. Since $x'^2 + y'^2 + 2x'y' . \cos x, y = r^2$, the above equation becomes on reduction

$$(x' + y' \cos x, y)\, x + (y' + x' \cos x, y)\, y = r^2.$$

Cor. 2. When the axes are rectangular, the equation just obtained becomes

$$y-y'=-\frac{x'}{y'}(x-x'),$$

$$\text{or } xx'+yy'=r^2.$$

66. **Prob. 2.** To draw a tangent to a circle from a given point (x', y') without it.

The circle being referred to rectangular axes, originating at the centre, let x, y be the co-ordinates of the point of contact,

$$\text{then } x^2+y^2=r^2 \ldots\ldots\ldots (1).$$

Again, since the point (x', y') is on the tangent passing through (x, y), we have

$$xx'+yy'=r^2 \ldots\ldots\ldots (2).$$

The co-ordinates may therefore be obtained by elimination between (1) and (2); and as the former of these equations is of the second degree, it follows, that there are two points of contact; or, in other words, that *two* tangents may be drawn to a circle from a given point without it.

It will be easier, however, instead of actually performing the elimination, to construct the *loci* of the above equations, since the points in which they intersect, will evidently be the points of contact.

Now the locus of (1) is the given circle; that of (2) is a straight line, which must therefore be the chord joining the points of contact, and which is thus determined.

$$\text{Let } x=0;\ \therefore y=\frac{r^2}{y'},$$

$$y=0;\ \therefore x=\frac{r^2}{x'}.$$

Hence, in CY, CX take CB and CA respectively equal to these values of y and x; join A, B, then the points P, p in which AB meets the circle, will be the points of contact required.

Cor. Hence may be proved the following theorem.

The chords joining the points of contact of each pair of tangents drawn from the several points of a line given in position, without a circle, will all meet in the same point.

Let QR be the given line, and let the circle be referred to rectangular co-ordinates originating at the centre, the axis of x being assumed perpendicular to QR (fig. 37.).

Then if QP, Qp be a pair of tangents drawn from any point $Q(x', y')$ we have, as was shewn above,

$$CA = \frac{r^2}{x'},$$

and as this value of CA is independent of y', it will be the same for all points whose abscissa is x', that is, for all points situated in the line QR. Whence the truth of the theorem.

67. Prob. 3. Through two given points to draw two straight lines that shall form a given angle $\tan^{-1} m$.

Let A, B be the given points, (fig. 38.) AP, BP the required lines, and let P be referred to rectangular axes AX, AY originating at A. From P drop PM perpendicular to AB, and let $AB=a$, $AM=x$, $MP=y$.

$$\text{Then } APB=\pi-(A+B);$$

$$\text{therefore } \tan APB= -\tan(A+B);$$

$$\text{that is, } -m=\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{\dfrac{y}{x}+\dfrac{y}{a-x}}{1-\dfrac{y^2}{ax-x^2}};$$

$$\therefore\ y^2+x^2-\frac{a}{m}y-ax=0,$$

which is the equation to a circle passing through the origin A, the co-ordinates of its centre being $\frac{a}{2}$, $\frac{a}{2m}$, and its radius $=\frac{a}{2m}\sqrt{(1+m^2)}$.

To trace the circle.

Let $y=0$; $\therefore x^2=ax$; $\therefore x=0$, and also $=a$.

The circle therefore passes through A and B.

$$\text{Let } x=0;\quad \therefore y^2=\frac{a}{m}y;\quad \therefore y=0, \text{ and } =\frac{a}{m}.$$

In AY take $AP'=\frac{a}{m}$, and the circle will pass through P'.

Hence lines drawn from A and B to any point in the segment APB, will have the property required.

Cor. Let APB be a right angle, then m being infinite, $\frac{a}{2m}$ vanishes, and therefore the locus of P is a semi-circle. According as APB is acute or obtuse, m will be positive or negative, and the centre will be situated *above* or *below* AB. The locus of P will therefore be an arc which is greater or less respectively, than a semi-circumference. (*Euc.* III. 31.)

68. Prob. 4. To find the co-ordinates of the points of intersection of two circles.

Let A and B be the centres of the two circles, r, r' their radii, and $AB=\delta$.

Then A being assumed, as the origin of the rectangular axes AX, AY, of which the former is supposed to pass through B,

The equation to the circle A is $x^2+y^2=r^2$ . . . . . . (1),

and that to the circle B is $(x-\delta)^2+y^2=r'^2$ . . (2).

At the point of intersection, these co-ordinates are identical; therefore subtracting the 2d equation from the 1st,

$$2\delta x - \delta^2 = r^2 - r'^2;$$

$$\therefore x = \frac{r^2 + \delta^2 - r'^2}{2\delta};$$

therefore by substition in (1),

$$y^2 = r^2 - \left(\frac{r^2 + \delta^2 - r'^2}{2\delta}\right)^2;$$

hence, by an operation similar to that in Art. 32. Cor. 2.

$$y = \pm \frac{1}{2\delta} \sqrt{\{(r + r' + \delta)(r + \delta - r')(r' + \delta - r)(r + r' - \delta)\}}.$$

Wherefore the circles cut one another in *two* points.

Since to the same abscissa there correspond two equal values of the ordinate, with opposite signs, it follows, that the chord *Pp* joining the points of section, is bisected at right angles by the line joining the centres.

Cor. 1. In order that the circles may cut another, these values of (y) must be *real*, or the quantity under the radical must be positive. Now this will be the case when all the factors are positive, or when two of them are positive and two negative; that is, since the first is necessarily positive, when any two of the last three factors are negative, and the remaining one positive. But by supposing any one of these factors negative, the other two will be found to be positive, thus

$$\text{let } r + \delta < r'; \; \therefore r < r' \text{ and } \delta < r';$$

$$\therefore (r' - r) + \delta \text{ and } (r' - \delta) + r \text{ are positive.}$$

Whence it follows, that each of the factors under the radical, must be positive, to produce an intersection of the circles.

Cor. 2. When the circles touch one another, the values of (y) are $=0$;

$$\therefore r+\delta-r'=0, \quad r'+\delta-r=0, \quad r+r'-\delta=0;$$

$$\therefore \delta=r \mp r'.$$

In other words, "two circles touch one another internally or externally, when the line joining their centres is equal to the difference or the sum of the radii."

69. Prob. 5. Two given circles being cut by a third, to find where the chords joining the points of intersection will meet.

Let A, B be the centres, (fig. 39.) r, r' the radii of two given circles which are cut by a third, whose centre is C, and radius ρ, in the points $a, m : b, n$; draw the chords $a\,m$, $b\,n$, and let them meet in P; required the position of this point.

The axes being supposed rectangular, let A be assumed as the origin, and AB as the axis of (x).

Let the co-ordinates of C be x' and y'; and $AB=\delta$.

Then the equation to the circle A is $\quad x^2+y^2=r^2 \ldots (1).$

that to $\quad B \quad (x-\delta)^2+y^2=r'^2 \ldots (2),$

and that to $\quad C \quad (x-x')^2+(y-y')^2=\rho^2 \ldots (3).$

Our object must be to find the equations to the chords am, bn, since it is only requisite to suppose x and y the same in both, in order to obtain the co-ordinates of the point in which those chords intersect.

Now when C cuts A, the co-ordinates at the points of intersection will be identical. Since x and y therefore are the same in (1) and (3), we have, subtracting the latter from the former,

$$2xx' + 2yy' = x'^2 + y'^2 + r^2 - \rho^2 \ldots . (4).$$

This being the equation to a straight line, it can be no other than the equation to the line am, which joins the points of intersection.

In like manner it may be shewn, that the equation to the chord bn is

$$2xx' + 2yy' - 2\delta x = r'^2 - \rho^2 - \delta^2 + x'^2 + y'^2 \ldots\ldots (5).$$

When am, bn intersect, these equations coincide; therefore subtracting (4) from (5), and dividing by 2δ, we have

$$x = \frac{r^2 - r'^2 + \delta^2}{2\delta}.$$

This value of (x) being constant, and independent of ρ, x', y', it follows that *whatever be the position and magnitude of the circle* C, *the point of intersection of all chords* am, bn *will lie in the line* MP, *which is at right angles to* AB.

Cor. 1. $BM = \delta - x = \dfrac{\delta^2 - r^2 + r'^2}{2\delta};$

$$\therefore BM - AM = \frac{r'^2 - r^2}{\delta} = \frac{r'^2 - r^2}{BM + AM};$$

$$\therefore BM^2 - AM^2 = r'^2 - r^2.$$

Hence, *the line AB is divided at the point M into two such parts that the difference of their squares is equal to the difference of the squares of the radii.*

Cor. 2. If from any point in MP, a tangent be drawn to each of the circles A, B, these tangents will be equal.

Let PQ, Pq be any two tangents so drawn; then

$$PQ^2 = Pn \,.\, Pb = Pm \,.\, Pa = Pq^2;$$

$$\therefore PQ = Pq.$$

Note. The point M is called the *radical centre*, and the line MP, the *radical axis*, of the two circles.

Cor. 3. When the circles A and B *cut* one another, the chord NP, (fig. 40.) coincides with the radical axis, (Art. 68.) Hence it follows, that *if three circles cut one another, the chords joining the points of intersection of each two of them, will meet in the same point.*

70. It may be proper, before quitting this part of the subject, to illustrate, by one or two examples, the utility of the equation to the circle, in the *demonstration of theorems.*

Ex. 1. If any two chords cut one another at right angles within a circle, the sum of the squares of the four segments is equal to the square of the diameter.

Let MN, PQ, be any two chords intersecting at right angles in O.

The equation to the circle when referred to rectangular axes, originating at O, is

$$(x-x')^2+(y-y')^2=r^2.$$

$$\text{Let } y=0;\ \therefore\ x=x'\pm\sqrt{(r^2-y'^2)}=OQ \text{ or } OP,$$

$$x=0;\ \therefore\ y=y'\pm\sqrt{(r^2-x'^2)}=OM \text{ or } ON;$$

$$\therefore\ OQ^2+OP^2+OM^2+ON^2$$

$$=2(x'^2-y'^2)+2r^2+2(y'^2-x'^2)+2r^2=4r^2=(\text{diameter})^2.$$

Ex. 2. Let P be any point in the diameter AB of a circle, MN any chord parallel to AB, required to prove that

$$PM^2+PN^2=PA^2+PB^2.$$

The point P being assumed as the origin of rectangular axes, the equation to the circle will be

$$(x-x')^2+y^2=r^2 \ldots\ldots\ldots\ldots (1).$$

$$\text{Let } y=0;\ \therefore\ x=x'\pm r=PB \text{ or } PA;$$

$$\therefore\ PA^2+PB^2=2(x'^2+r^2) \ldots\ldots (2).$$

Now the equation to the chord MN, which is parallel to AB, is $y = m$; therefore eliminating y from equation (1), we have for the abscissas of M and N,

$$x = x' \pm \sqrt{(r^2 - m^2)},$$

and these points having the same ordinate (m),

$$\begin{aligned} PM^2 + PN^2 &= 2m^2 + 2x'^2 + 2r^2 - 2m^2 \\ &= 2(x'^2 + r^2) \\ &= PA^2 + PB^2 \text{ by (2).} \end{aligned}$$

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CHAP. VI.

ON THE TRANSFORMATION OF CO-ORDINATES.

71. By means of the equation to a curve, we are enabled to define its nature and investigate its properties. These two objects, therefore, will be most easily attained, when the equation is reduced to its simplest form. Now, as the form of an equation depends on the position and direction of the axes, it can only be modified by transferring the curve to a different system of co-ordinates. This operation while it tends to simplify the equation, does not affect the *nature* of the curve, which is quite independent of the axes to which it is referred.

Thus the equation to the circle is obtained in its most general form, by referring the curve to oblique axes, originating at any assumed point; it is then reduced, first by supposing the axes rectangular, and next, by assigning different positions to the origin. The *nature* of the circle remains in all cases unchanged, while the investigation of its properties is facilitated, in proportion as the form of its equation is simplified.

The operation of displacing the origin, or changing the direction, of the axes, or of altering at once both their origin and direction, is called the *transformation of co-ordinates.*

72. Prob. 1. To transform the equation to any proposed curve from one system of oblique axes to another; the origin being the same.

Let AX, AY, be the primitive, AX', AY', (fig. 41.) the new axes, x, y the co-ordinates of a point referred to the former, x', y' those of the same point when referred to the latter.

Since the nature of a curve is independent of any particular system of axes, the object of the problem is to determine how x and y must be expressed in terms of x' and y', in order that, when the latter is substituted for the former in a given equation between x and y, the degree of the resulting equation may remain unaltered. This, it is evident, can only take place, when the co-ordinates of the one system are *linear functions* of those of the other.

For example, let the proposed equation be that to the *straight line* or *circle*, then, in order that the transformed equations may continue to be of the *first* or *second* degree, respectively, x' and y' must necessarily be linear functions of x and y.

We shall assume, therefore, generally, that the relation between the primitive and the new co-ordinates may be thus expressed,

$$x = mx' + ny' \text{ and } y = m'x' + n'y' \ldots\ldots (1),$$

m, n; m', n' being constant quantities, which are now to be determined.

(1) Let $y' = 0$; therefore the point will be situated on AX' as at P; draw PM parallel to AY.

$$\text{Then } m = \frac{x}{x'} = \frac{AM}{AP} = \frac{\sin x', y}{\sin x, y},$$

$$m' = \frac{y}{x'} = \frac{PM}{AP} = \frac{\sin x', x}{\sin x, y}.$$

(2) In like manner, by supposing $x' = 0$, we obtain

$$n = \frac{\sin y', y}{\sin x, y}, \quad n' = \frac{\sin y', x}{\sin x, y}.$$

Therefore, m, m'; n, n', being replaced by the values just obtained, equation (1) becomes

$$x = \frac{1}{\sin x, y}\{x' \sin x', y + y' \sin y', y\},$$

$$y = \frac{1}{\sin x, y}\{x' \sin x', x + y' \sin y', x\}.$$

Hence, any equation between x and y may, by substitution of these values, be transformed to an equation between x' and y': as was required.

73. **Cor. 1.** Let the primitive axes be rectangular, and the new ones oblique: then, $\sin x, y = 1$.

$$\text{Also } \sin x' y = \sin\left(\frac{\pi}{2} - x', x\right) = \cos x', x,$$

$$\sin y', y = \sin\left(\frac{\pi}{2} + y', x\right) = \cos y', x;$$

therefore, the above formulas become

$$x = x' \cos x', x + y' \cos y', x,$$

$$y = x' \sin x', x + y' \sin y', x.$$

74. **Cor. 2.** Let both systems be rectangular; therefore, the formulas become

$$x = x' \cos x', x - y' \sin x', x,$$

$$y = x' \sin x', x + y' \cos x', x.$$

75. **Cor. 3.** Let the primitive axes be oblique, and the new ones rectangular.

$$\text{Then, } \sin y', y = \cos x', y, \text{ and } \sin y', x = \cos x', x;$$

therefore, we have

$$x = \frac{1}{\sin x, y}\{x' \sin x', y + y' \cos x', y\},$$

$$y = \frac{1}{\sin x, y}\{x' \sin x', x + y' \cos x', x\}.$$

76. If, besides changing the *direction* of the axes, it be required at the same time to displace their *origin*, we have only in the foregoing formulas to add the new abscissa to the value of x, and the new ordinate to the value of y.

Thus, if a, b, be the co-ordinates of the new origin, then the general formulas will become

$$x = a + \frac{1}{\sin x, y} \{x' \sin x', y + y' \sin y', y\}$$

$$y = b + \frac{1}{\sin x, y} \{x' \sin x' x + y' \sin y', x\}$$

When the origin alone is displaced, and the new axes supposed to remain parallel to the primitive ones, then the formulas are in this case,

$$x = a + x'$$

$$y = b + y'$$

77. **Prob. 2.** To pass from one system to another, of *polar* co-ordinates.

Let a, b, be the rectilinear co-ordinates of the new pole S, and let the axis to which the radius vector is referred make with the primitive axis an angle α: then, the formulas of Art. 46. become, by substitution,

$$x = a + r \cdot \frac{\sin \{x, y - (\omega + \alpha)\}}{\sin x, y}$$

$$y = b + r \cdot \frac{\sin (\omega + \alpha)}{\sin x, y}$$

78. **Cor.** Let the axis to which r is referred be parallel to AX, then $\alpha = 0$, and the formulas become

$$x = a + r \cdot \frac{\sin (x, y - \omega)}{\sin x, y}$$

$$y = b + r \cdot \frac{\sin \omega}{\sin x, y}$$

By adopting the usual notation, these formulas may be thus expressed,

$$x = a + r \, . \, \frac{\sin r, y}{\sin x, y}$$

$$y = b + r \, . \, \frac{\sin r, x}{\sin x, y}$$

79. Let the axes be rectangular, then the formulas in Art. 77 and 78, become

$$\left.\begin{array}{l} x = a + r \cos(\omega + a) \\ y = b + r \sin(\omega + a) \end{array}\right\} \ldots\ldots (1),$$

and

$$\left.\begin{array}{l} x = a + r \cos \omega \\ y = b + r \sin \omega \end{array}\right\} \ldots\ldots\ldots\ldots (2).$$

80. By supposing the angle which the radius vector makes with the axis, to pass through all degrees of magnitude between 0 and 360^0, the signs of the trigonometrical functions of that angle will determine the *position* of the point *P*. Hence, in the application of the foregoing formulas, the radius vector (r) must always be considered as *positive;* and those values of (r), which are negative, must be rejected. See *Scholium*, p. 35.

SECTION III.

ON THE DISCUSSION AND PROPERTIES OF LINES OF THE SECOND ORDER.

CHAP. I.

ON LINES OF THE SECOND ORDER IN GENERAL.

81. A CURVE, whose equation is of the n^{th} degree, is called a *line of the n^{th} order.*

Agreeably to this definition, the straight line, since it is characterized by an equation of the first degree, is a line of the *first order,* and the circle, for a similar reason, is a line of the *second order.*

82. PROP. 1. To find the locus of the equation of the second degree between two variables.

The general form of this equation is

$$ay^2+bxy+cx^2+dy+ex+f=0,$$

in which a, b, c ... are constant quantities that may be positive or negative, fractional or integral.

The equation being resolved in terms of x and y successively, we have

$$y=-\frac{bx+d}{2a}\pm\frac{1}{2a}\sqrt{\{(b^2-4ac)x^2+2(bd-2ae)x+d^2-4af\}},$$

$$x=-\frac{by+e}{2c}\pm\frac{1}{2c}\sqrt{\{(b^2-4ac)y^2+2(be-2cd)y+e^2-4cf\}}.$$

Let the known coefficients in the irrational part of these equations be represented by single lettters, and let

$$-\frac{b}{2a}=m,\quad -\frac{d}{2a}=n;\quad -\frac{b}{2c}=m',\quad -\frac{e}{2c}=n'.$$

Then, by substitution,

$$y=mx+n\pm\frac{1}{2a}\sqrt{\{Ax^2+2Bx+C\}}\ldots\ldots(1),$$

$$x=m'y+n'\pm\frac{1}{2c}\sqrt{\{A'y^2+2B'y+C'\}}\ldots\ldots(2).$$

1st. To construct the values of (y).

This value consists of two parts, of which the rational part, $mx+n$, is the equation to a straight line, and may therefore be constructed by (51). Let BR, (fig. 42.) be this line, the axes AX, AY, being inclined at any angle whatever. In AX, take any point M, and draw MP parallel to AY, meeting BR in N.

Let $AM=x$, then MN will $=mx+n$.

Again, take NP, Np, each $=\frac{1}{2a}\sqrt{\{Ax^2+2Bx+C\}}$,

then MP, Mp, will be the values of (y) required.

For $MP=MN+NP=mx+n+\frac{1}{2a}\sqrt{\{Ax^2+2Bx+C\}}$,

$Mp=MN-Np=mx+n-\frac{1}{2a}\sqrt{\{Ax^2+2Bx+C\}}$.

2dly. To construct the values of x.

Let DS (fig. 43.) represent the line $m'y+n'$, take $AM'=y$, draw $M'Q$ parallel to AX, meeting DS in N', then, assume $N'Q$, $N'q$, each

$$=\frac{1}{2c}\sqrt{\{Ay^2+2B'y+C'\}},$$

and it may be shewn, as before, that $M'Q$, $M'q$ are the two values of x.

The curves, which are the loci of the above equation, are termed *lines of the second order.*

83. Prop. 2. A straight line cannot cut a line of the second order in more than two points.

Let the curve be cut by a straight line, whose equation is $y=mx+n$; then, at the points of intersection, the co-ordinates of the line and of the curve will be identical. Substituting, therefore, the value of (y) in the general equation, we have

$$a(mx+n)^2+bx(mx+n)+cx^2+d(mx+n)+ex+f=0,$$

which, on reduction, becomes

$$\{am^2+bm+c\}\,x^2+\{(2am+b)n+dm+e\}\,x+an^2+dn+f=0.$$

This equation, being of the second degree, can have only *two* roots, which, when real, represent the abscissas of the points of intersection. Hence, the truth of the proposition.

84. Prop. 3. To find the locus of the middle points of any number of parallel chords.

$$\text{Let } y=mx\ldots\ldots\ldots\ldots(a)$$

be the equation to any line APp drawn through the origin, and cutting the curve in the points P, p (fig. 44): take O (x', y'), the middle point of any chord Qq, parallel to APp; then, the question is to find the relation between the co-ordinates of O.

If the origin be transferred to O, the equation to the curve will become

$$a(y+y')^2+b(x+x')(y+y')+c(x+x')^2+d(y+y')+e(x+x')+f=0,$$

and the equation to Qq, will become $y = mx$.

Now the points in which Qq intersects the curve will be determined by supposing the co-ordinates in the last two equations identical, whence

$$a(mx+y')^2+b(x+x')(mx+y')+c(x+x')^2+d(mx+y')+e(x+x')+f=0.$$

But Qq being bisected in O, the roots of this equation will be equal, and the coefficient of the second term will, therefore, vanish; whence, collecting the terms involving x, we have

$$2amy' + by' + bmx' + 2cx' + dm + e = 0,$$

or, arranging, and suppressing the accents,

$$(2am+b)\,y + (bm+2c)\,x + dm + e = 0 \dots\dots\dots\dots (1),$$

the equation to *a straight line*, which is, therefore, the locus required.

The straight line which has thus been shewn to bisect a system of parallel chords is called a *diameter;* and the points in which it meets the curve are called the *vertices*.

COR. The diameter corresponding to any other chord

$$y = m'x \dots\dots\dots\dots\dots (\beta),$$

will have for its equation

$$(2am'+b)\,x + (bm'+2c)\,y + dm' + e = 0 \dots\dots\dots (2).$$

85. PROP. 4. If either of these diameters be parallel to the corresponding chords of the other, then, reciprocally, the latter will be parallel to the corresponding chords of the former.

Let the diameter (1), for instance, be parallel to the chord (β); we are to prove that the diameter (2) will be parallel to the chord (α).

Since (1) is parallel to (β), we have (Art. 57. Cor. 3.)

$$m' = -\frac{bm+2c}{2am+b};$$

$$\text{therefore } m = -\frac{bm'+2c}{2am'+b};$$

that is, the chord (α) is parallel to the diameter (2).

Hence it appears that *each of the diameters* (1) *and* (2) *bisects the chords drawn parallel to the other.*

Diameters thus related to each other are called *conjugate diameters.*

86. Cor. If the equation to any diameter be

$$y = mx + n,$$

the equation to the diameter conjugate to it, will be

$$y = -\frac{bm+2c}{2am+b}x + n'.$$

The number of systems of conjugate diameters is evidently unlimited.

87. Prop. 5. To determine whether any system of conjugate diameters can be rectangular.

$$\text{Let } y = mx + n,$$

$$\text{and } y = m'x + n'$$

be any system of conjugate diameters;

$$\text{then (86) } m' = -\frac{bm+2c}{2am+b},$$

or, multiplying out, and dividing each term by $2a$,

$$mm' + \frac{b}{2a}(m+m') + \frac{c}{a} = 0 \ldots\ldots\ldots (1).$$

Also, since the diameters are by hypothesis rectangular, we have (Art. 57. Cor. 1.)

$$m' = -\frac{1 + m \cos \gamma}{m + \cos \gamma},$$

(γ) being the inclination of the axes;

$$\therefore\ mm' + (m + m') \cos \gamma + 1 = 0 \ldots\ldots\ldots (2).$$

Wherefore m and m' may be found by means of equations (1) and (2).

Now the quadratic equations which serve to determine m and m' will be identical, as is sufficiently evident from the symmetrical form of (1) and (2): consequently, m and m' being the two roots of the same quadratic equation, there can be but *one* system of conjugate diameters that are rectangular.

These are called the *principal diameters*, or the *axes*, and the points in which they meet the curve are called the *principal vertices*.

88. Prop. 6. To find the co-ordinates of the point of intersection of any two diameters.

Let the equations to these diameters be

$$(2am + b)\, y + (bm + 2c)\, x + dm + e = 0,$$

$$\text{and } (2am' + b)\, y + (bm' + 2c)\, x + dm' + e = 0.$$

Now at the point of intersection the co-ordinates are identical; therefore subtracting the second equation from the first, we have

$$2a\,(m - m')\, y + b\,(m - m')\, x + d\,(m - m') = 0,$$

$$\text{or dividing by } m - m',\ 2ay + bx + d = 0 \ldots\ldots (1).$$

Again, subtracting the first equation when multiplied by m', from the second, when multiplied by m, we have

$$b\,(m - m')\, y + 2c\,(m - m')\, x + e\,(m - m') = 0,$$

$$\text{or } by + 2cx + e = 0 \ldots\ldots\ldots (2).$$

Hence, equating the values of (y), deduced from (1) and (2),

$$-\frac{bx+d}{2a}=-\frac{2cx+e}{b};$$

$$\therefore\ x=\frac{2ae-bd}{b^2-4ac},$$

and, similarly, $y=\dfrac{2cd-be}{b^2-4ac}$,

which are the co-ordinates required of the point of intersection.

This point is called the *centre.*

Hence the centre may be thus found.

Draw any two parallel chords, and bisect each of them: do the same with any other two parallel chords, then the centre required will be the point in which the lines joining the middle points of each pair of chords intersect.

89. The denominator b^2-4ac may be either finite or $=0$; in the former case the curve has a centre, in the latter, it has not a centre, and all the diameters are consequently parallel.

Lines of the second order may therefore be distributed into two classes, those, namely, which have, and those which have not, a centre.

Again, since b^2-4ac when finite, may be either positive or negative, the lines included in the first class may be subdivided into two species, corresponding to this difference of sign.

The general equation of the second degree comprises, therefore, three distinct curves, which are named as follows:

(1) An *ellipse,* when b^2-4ac is *negative.*

(2) An *hyperbola,* when b^2-4ac is *positive.*

(3) A *parabola,* when $b^2-4ac=0$.

CHAP. II.

ON THE ELLIPSE AND HYPERBOLA.

90. **Prop. 1.** To find the form of the equation to the ellipse and hyperbola, when the axes of the co-ordinates are parallel to any system of conjugate diameters.

The equation to any diameter is

$$(2am+b)\,y+(bm+2c)\,x+dm+e=0,$$

$$\text{in which } m=\frac{\sin\rho,x}{\sin\rho,y}.$$

Now let $\sin\rho, x=0$, that is, let the diameter be parallel to the axis of x, then $m=0$, and the equation becomes

$$by+2cx+e=0\ldots\ldots\ldots\ldots(1).$$

Next, let $\sin\rho, y=0$, that is, let the diameter be parallel to the axis of y, then $m=\infty$, and the equation becomes

$$2ay+bx+d=0\ldots\ldots\ldots\ldots(2).$$

Hence, when the axes are parallel to a system of conjugate diameters, of these two equations the first will bisect the chords parallel to the axis of x, and ought, therefore, to involve x alone (Art. 46.); and the second, for a similar reason, ought to involve only y. It follows, therefore, that $b=0$, or that *the coefficient of xy vanishes.* Whence the general equation becomes

$$ay^2 + cx^2 + dy + ex + f = 0,$$

which is of the form required.

91. Cor. 1. To find the form of the equation when the centre is the origin.

In this case, the equations to the diameters (1) and (2) will be of the form

$$y = ax \text{ (Art. 49. Cor. 3.);}$$

therefore, $d = 0$ and $e = 0$: that is, *the coefficients of x and y vanish.*

Whence the general equation becomes

$$ay^2 + bxy + cx^2 + f = 0.$$

When the curve is referred to conjugate diameters, the form of the equation is

$$ay^2 + cx^2 + f = 0.$$

92. Cor. 2. Since $b^2 - 4ac$ is negative in the ellipse, and positive in the hyperbola, it follows, that when $b = 0$, a and c must have the same sign in the former, and different signs in the latter.

93. Prob. 2. To find the equation to the ellipse and hyperbola, when referred to their principal diameters.

Let CA, CB, be the principal semi-diameters, of which the former is supposed to be the axis of x, the latter that of y.

Assume $CA = b$, $CB = b$.

Then, the form of the required equation will (91) and (92) be

$$Ay^2 \pm Cx^2 + f = 0.$$

Hence, (1) in the ellipse, (fig. 45.)

Let $y=0$; $\therefore x^2 = -\frac{f}{C} = CA^2 = a^2$; $\therefore C = -\frac{f}{a^2}$,

$x=0$; $\therefore y^2 = -\frac{f}{A} = CB^2 = b^2$; $\therefore A = -\frac{f}{b^2}$;

therefore by substitution the equation becomes

$$-\frac{f}{b^2}y^2 - \frac{f}{a^2}x^2 = -f,$$

and dividing by f,

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1 \ldots\ldots\ldots (1),$$

or multiplying by a^2b^2,

$$a^2y^2 + b^2x^2 = a^2b^2 \ldots\ldots (2).$$

(2) In the hyperbola.

Since the coefficients of y^2 and x^2 must have different signs, (92), the equation required will be either

$$a^2y^2 - b^2x^2 = -a^2b^2,$$

$$\text{or } b^2x^2 - a^2y^2 = -a^2b^2.$$

These equations, when divided by a^2b^2, become

$$\left.\begin{array}{l} \frac{y^2}{b^2} - \frac{x^2}{a^2} = -1 \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \end{array}\right\} \ldots\ldots\ldots\ldots (3).$$

The equations marked (1) and (3) are of a form analogous to that of the equation to a straight line in Art. 49. Cor. 4.

94. Cor. 1. Let the origin be transferred to A, one of the principal vertices, then if $AM = x'$, we have $x = a - x'$, therefore, the equation becomes

$$y^2 = \frac{b^2}{a^2}(2ax' - x'^2),$$

or suppressing the accents, which were used merely to distinguish the one abscissa from the other,

$$y^2 = \frac{b^2}{a^2}(2ax - x^2).$$

In like manner the equation to the hyperbola becomes

$$y^2 = \frac{b^2}{a^2}(2ax + x^2).$$

95. Cor. 2. Let $a = b$, then the two equations to the ellipse become

$$y^2 = a^2 - x^2, \text{ and } y^2 = 2ax - x^2,$$

which represent a circle.

The equations to the hyperbola become

$$y^2 = x^2 - a^2, \quad \text{and} \quad y^2 = 2ax + x^2.$$

The hyperbola which these equations represent, is called *equilateral*, and bears to the common hyperbola a relation analogous to that which the circle bears to the ellipse.

96. Prob. 3. To trace the figure of the ellipse and of the hyperbola from their equations.

The rectangular axes being CX, CY, let it be required

(1) To trace the ellipse.

Resolving the equation in terms of x, we have

$$y = \pm\frac{b}{a}\sqrt{(a^2 - x^2)}.$$

Let $y = 0$; $\therefore x = \pm a = CA$ or CV.

$x = 0$; $\therefore y = \pm b = CB$ or Cb, When (x) is positive, and increases from 0 to (a), y is real, and diminishes from b to 0. In this case, each value of (x) furnishes *two* equal values of y with contrary signs.

When $x > a$, the values of y are imaginary, and therefore the curve does not extend beyond A, to the right.

If x be negative, x^2 is positive, and it may be shewn, as before, that the curve does not extend beyond V to the left.

Again, the equation being resolved in terms of y,

$$x = \pm \frac{a}{b} \sqrt{(b^2 - y^2)},$$

and it may be proved exactly in the same manner as in the former case that the curve is not continued beyond B or b. The curve therefore is wholly comprised between the parallels PM, QN and PQ, MN.

But a straight line, it is known, (Art. 83.) cannot cut an ellipse in more than *two* points. Hence the ellipse must have the form assigned to it in fig. 45.

(2) To trace the hyperbola (fig. 46.)

Resolving the first of the equations (3) in terms of (x), we have

$$y = \pm \frac{b}{a} \sqrt{(x^2 - a^2)}$$

$$\text{Let } y = 0; \therefore x = \pm a = CA \text{ or } CV,$$

$$x = 0; \therefore y = \pm b \sqrt{(-1)}.$$

These values of (y) being imaginary, the curve cannot intersect the axis CY. To maintain the analogy, however, between the ellipse and hyperbola, we assume CB and $Cb = b$.

Let $x < a$; therefore the values of (y) being still imaginary, no part of the curve can lie between C and A.

Let $x > a$; therefore the values of (y) are real, and to each assumed value of x, there correspond *two* equal values of y, with contrary signs.

In proportion as x increases, the values of (y) are increased. When x therefore is assumed indefinitely great, the values of (y) become indefinitely great also. Hence to the right of C there is an infinite branch ZAz.

If x be supposed negative, it may in like manner be shewn, that to the left there is an infinite branch $Z'Vz'$.

The hyperbola therefore consists of two infinite branches, symmetrically situated on opposite sides of the centre.

Again, resolving the second of equations (3) in terms of (y), we have (fig. 47.)

$$x = \pm\frac{a}{b}\sqrt{(y^2 - b^2)}.$$

$$\text{Let } x = 0; \therefore y = \pm b = CB \text{ or } Cb,$$

$$y = 0; \therefore x = \pm a\sqrt{(-1)}.$$

These values of x being imaginary, the curve cannot meet the axis CX. We assume, however, as before, CA and $CV = a$.

Let $y < b$; therefore the values of y being still imaginary, no part of the curve can lie between C and B.

The investigation being continued as before, we conclude as in the former case, that the hyperbola is composed of two infinite branches NBn and $N'bn'$.

Figure (46) corresponds to $a^2y^2 - b^2x^2 = -a^2b^2$,

(47) to $b^2x^2 - a^2y^2 = -a^2b^2$.

Observation. In investigating the properties of the hyperbola, we usually employ the former of these equations,

$$a^2y^2 - b^2x^2 = -a^2b^2,$$

which is derived from the equation to the ellipse, by merely changing the sign of b^2.

97. PROB. 4. To find the distance of any point in the ellipse and hyperbola from the centre.

Let ρ be the distance sought.

Then (1) in the ellipse,

$$\rho = \sqrt{(x^2+y^2)} = \sqrt{\left\{x^2 + \frac{b^2}{a^2}(a^2-x^2)\right\}}$$

$$= \sqrt{\left\{b^2 + \frac{a^2-b^2}{a^2}x^2\right\}}.$$

The greater x is, the greater will ρ be; let x be the greatest possible, that is, let it $=a$, then

$$\rho = \pm a = CA \text{ or } CV.$$

In proportion as x decreases, ρ will decrease; let x be the least possible, that is, let it $=0$, then

$$\rho = \pm b = CB \text{ or } Cb.$$

Hence, (fig. 45.) CA is the greatest, and CB the least distance from the centre; in other words, AV is the greatest, and Bb the least diameter of the ellipse.

(2) It may be shewn in a similar manner, that AV is the greatest, and Bb the least, diameter of the hyperbola.

The diameters AV, Bb in both curves, are called respectively the *major* and *minor axes*.

98. The equation to a circle described on the major axis of an ellipse is $y^2 = a^2 - x^2$. If therefore MQ be an ordinate to the circle meeting the ellipse in P, we have (fig. 48.)

$$MP = \frac{b}{a}\sqrt{(a^2-x^2)},$$

$$\text{and } MQ = \sqrt{(a^2-x^2)};$$

$$\therefore \frac{MP}{MQ} = \frac{b}{a}.$$

Hence is derived an easy method of describing the ellipse by points.

From any point M in the major axis, draw an ordinate MQ to the circle, join CQ, cutting the circle described on Bb in q; through q draw qP parallel to AV, meeting MQ in P; then P will be a point in the ellipse.

$$\text{For } MP = MQ \cdot \frac{Cq}{CQ} = MQ \cdot \frac{b}{a},$$

as it ought to be. If Mp be taken equal MP, then will p also be a point in the ellipse. In like manner, two other points may be determined, equidistant from C to the left.

This process is not applicable to the hyperbola, since its equation can only be compared with the equation $y^2 = x^2 - a^2$, which represents an equilateral hyperbola. The mode of constructing the latter must therefore be known, before we can employ it in the construction of the common hyperbola.

99. **Prob. 5.** To find the equations to the ellipse and hyperbola, when referred to a given system of conjugate diameters, the centre being the origin.

Let CP, CD, (fig. 49.) be any two semi-conjugate diameters, of which the former is supposed to be the axis of x, the latter that of y.

$$\text{Assume } CP = a', \ CD = b'.$$

Then it may be proved precisely as in Art. 93, that the required equations are

$$a'^2y^2 \pm b'^2x^2 = \pm a'^2b'^2,$$

the upper sign being used for the ellipse, the lower for the hyperbola.

Observation. The equation to the hyperbola is deduced from that to the ellipse, by changing, as in Art. 93, the sign of b'^2. When CP, CD are called conjugate diameters in the hyperbola, it is to be understood, that they are so called only by analogy, since CD does not, in fact, meet the curve.

100. Cor. 1. Hence $y^2 = \frac{b'^2}{a'^2}(2a'x \mp x^2)$;

that is, if any ordinate Qv be drawn in the ellipse or hyperbola, and PC be produced to meet the curve in G,

$$Pv \,.\, vG : Qv^2 :: PC^2 : CD^2.$$

When the diameters PC, CD are rectangular, the above proportion becomes

$$AM \,.\, MV : MP^2 :: AC^2 : BC^2.$$

101. Cor. 2. Since $y^2 = \frac{b'^2}{a'^2}(2a'x \mp x^2)$

$$= 2\frac{b'^2}{a'}x \mp \frac{b'^2}{a'^2}x^2,$$

if $\frac{b'^2}{a'} = p$; we have $y^2 = 2px \mp \frac{p}{a'}x^2$.

The line $2p$ which has thus been assumed a third proportional to any two conjugate diameters, is called the *parameter* to the diameter PG. When the axes are rectangular, then $2p$ is called the *principal parameter*, or the *latus rectum*.

102. Cor. 3. If $y = mx$ be the equation to any diameter, then (86) the equation to the diameter conjugate to it will be either

$$y = \mp \frac{b'^2}{a'^2} \cdot \frac{1}{m} x,$$

$$\text{or } y = \mp \frac{b^2}{a^2} \cdot \frac{1}{m} \cdot x,$$

according as the curve is referred to *any* system of conjugate diameters, or to the axes.

103. Prob. 6. To find the co-ordinates of the points, in which any diameter intersects the curve.

Let $y = mx$ be the equation to any diameter,

$a'^2y^2 \pm b'^2x^2 = a'^2b'^2$ that to the curve;

then, the co-ordinates required will be obtained by elimination between these two equations.

Hence,

(1) in the ellipse, $x = \dfrac{a'b'}{\sqrt{(a'^2m^2 + b'^2)}}$, and $y = \dfrac{ma'b'}{\sqrt{(a'^2m^2 + b'^2)}}$.

(2) in the hyperbola, $x = \dfrac{a'b'}{\sqrt{(b'^2 - a'^2m^2)}}$, and $y = \dfrac{ma'b'}{\sqrt{(b'^2 - a'^2m^2)}}$.

It appears, therefore, that in the ellipse, any diameter whatever will always meet the curve, and that in the hyperbola, a diameter can meet the curve only when $b'^2 - a'^2m^2$ is positive, or, $m < \dfrac{b'}{a'}$.

If $m > \dfrac{b'}{a'}$, the diameter never meets the curve,

and if $m = \dfrac{b'}{a'}$, the diameter meets the curve only at a point infinitely distant.

Let Pp, Dd, be any two conjugate diameters, then, if Qq be drawn through P, parallel and equal to Dd, and CQ, Cq be joined, and produced to Z, z the diameters CZ, Cz will intersect the hyperbola at an infinite distance, and will separate those diameters which meet, from those which can never meet, the curve.

The lines CZ, Cz are called *asymptotes*, and are defined by the equation $y = \pm \dfrac{b'}{a'}x$.

104. **Cor.** When the abscissa is supposed to be indefinitely great, it follows that the ordinate to the hyperbola must equal the ordinate to the asymptote.

In fact, $y = \pm \dfrac{b'}{a'}\sqrt{(x^2 - a'^2)}$

$$= \pm \frac{b'}{a'} x \sqrt{\left(1 - \frac{a'^2}{x^2}\right)}$$

$$= \therefore \pm \frac{b'}{a'} x,$$

when x is infinitely great.

105. *Hence, if the equation to a curve be reducible to the form*

$$y = ax + b + \frac{c}{x} + \frac{d}{x^2} + \ldots\ldots$$

the straight line $y = ax + b$ is an asymptote to the curve.

For if x be assumed indefinitely great, then $\frac{c}{x} + \frac{d}{x^2} + \ldots$ is indefinitely small, and, therefore, the ordinate to the curve becomes equal to the ordinate of the straight line $y = ax + b$.

106. It has already been remarked, that the equation to the ellipse becomes the equation to the hyperbola, when the sign of b^2 or of b'^2 in the former curve is changed. From the close analogy thus subsisting between these curves, it is evident, that the properties of the one must be deducible immediately from those of the other. For the sake of brevity, therefore, we shall, in what follows, comprehend both curves in the enunciation of any proposition, and shall deem it sufficient to give a demonstration in the case of the ellipse, which occurs the more frequently of the two in mathematical inquiries.

Any peculiarity that presents itself either in the nature or proof of the proposition, when applied to the hyperbola, shall be noted and explained.

CHAP. III.

ON THE PROPERTIES OF TANGENTS TO THE ELLIPSE AND HYPERBOLA.

107. **Def.** A TANGENT to a line of the second order is a straight line that meets the curve, and being produced, does not cut it. That part of the axis, intercepted between the foot of the ordinate, and the point where the tangent meets the axis is called the *sub-tangent.*

Hence, if a straight line be drawn cutting the curve, and the two points of section be then supposed to coincide, the secant will become a tangent. It is on this principle that we shall resolve the following problem.

108. **Prop. 1.** To draw a tangent at a given point (x', y'), in the ellipse and hyperbola.

The equation to a secant drawn through this point, is

$$y-y'=m(x-x')\ldots\ldots(1),$$

(m) being indeterminate.

Now, as x', y', are the co-ordinates of a point in the curve, they will satisfy the general equation

$$a'^2y^2 \pm b'^2x^2 = \pm a'^2b'^2;$$

we have, therefore,

$$a'^2y'^2 \pm b'^2x'^2 = \pm a'^2b'^2;$$

and subtracting the lower from the upper equation,

$$a'^2(y^2-y'^2) = \mp b'^2(x^2-x'^2),$$

$$\text{or } a'^2(y+y')(y-y') = \mp b'^2(x+x')(x-x');$$

hence, replacing $y-y'$ by its value in (1), and dividing the result by $x-x'$, we have

$$a'^2 m(y+y') = \mp b'^2(x+x').$$

If now the points of the section be supposed to coincide, $x=x'$, and $y=y'$, and the secant becomes a tangent;

$$\therefore\ m = \mp \frac{b'^2}{a'^2}\,\frac{x'}{y'};$$

this value of (m) being substituted in (1), the equation to the tangent becomes

$$y-y' = \mp \frac{b'^2}{a'^2}\cdot\frac{x'}{y'}(x-x')\ldots\ldots(1),$$

the upper sign being used for the ellipse, the lower for the hyperbola.

109. Cor. 1. The equation to the tangent just deduced becomes, on reduction,

$$a'^2yy' \pm b'^2x'x = \pm a'^2b'^2.$$

This form of the equation is the one most frequently employed.

If $y=y'$, and $x=x'$, we again fall upon the general equation to the ellipse and hyperbola.

110. Cor. 2. It appears, by referring back to Art. 102, that the tangent, drawn at the extremity of any diameter, is parallel to the corresponding conjugate diameter.

Hence, in the ellipse, if tangents be drawn at the extremities of any two conjugate diameters, Pp, Dd (fig. 51.), a parallelogram Nn will thus be formed, which circumscribes the ellipse.

In the hyperbola, there can be only two tangents drawn, since the conjugate diameter Dd does not meet the curve, (fig. 52.).

111. In finding the equation to the tangent, we have assumed as axes any system whatever of conjugate diameters. It is frequently convenient, however, to employ rectangular co-ordinates, in which case, the equation will have the same form as before, the conjugate diameters $2a'$ and $2b'$ being merely replaced by the axes of the curve. The equation will therefore be

$$a^2yy' \pm b^2xx' = \pm a^2b^2.$$

Def. If from the point of contact P, a line be drawn at right angles to the tangent, meeting the major axis in G, then PG is called the *normal*, and GM the *sub-normal*.

112. Prop. 2. To find the equation to the normal.

The normal being drawn through the point (x', y'), at right angles to the tangent, its equation will be, by Prop. 1,

$$y - y' = \pm \frac{a^2}{b^2} \cdot \frac{y'}{x'} (x - x'),$$

the upper sign for the ellipse, the lower for the hyperbola.

113. Cor. Let $y = 0$;

$$\therefore \; x - x' = GM = \mp \frac{b^2}{a^2} \cdot \frac{y'x'}{y'} = \mp \frac{b^2}{a^2} x',$$

which is the value of the sub-normal.

Hence, the normal

$$= \sqrt{\{(x-x')^2 + y'^2\}} = \sqrt{\left(y'^2 + \frac{b^4}{a^4} x'^2\right)}.$$

$$= \sqrt{\left\{\frac{b^2}{a^2}\cdot(a^2 - x^2) + \frac{b^4}{a^4}x'^2\right\}}$$

$$= \pm\frac{b}{a}\sqrt{\left\{a^2 - \frac{a^2 - b^2}{a^2}x'^2\right\}}\ldots\ldots(1),\ \text{in the ellipse,}$$

$$\text{and } = \pm\frac{b}{a}\sqrt{\left\{\frac{a^2 + b^2}{a^2}x'^2 - a^2\right\}}\ldots\ldots(2),\ \text{in the hyperbola.}$$

114. Prop. 3. If a tangent at any point Q in the ellipse or hyperbola meet any semi-conjugate diameters CP, CD in T, t and QM, Qm be parallel to CD, CP, it is required to prove, that (fig. 53.)

$$CT\,.\,CM = CP^2,$$

$$\text{and } Ct\,.\,Cm = CD^2.$$

The general equation to the tangent being

$$a'^2yy' \pm b'^2xx' = \pm a'^2b'^2.$$

Let $y = 0$; $\therefore\ x = CT = \pm\frac{a'^2}{x'} = \frac{CP^2}{CM}$; $\therefore\ CT\,.\,CM = CP^2$;

$x = 0$; $\therefore\ y = Ct = \pm\frac{b'^2}{y'} = \frac{CD^2}{Cm}$; $\therefore\ Ct\,.\,Cm = CD^2$.

Cor. 1. When the axes are rectangular,

$$CT\,.\,CM = CA^2, \text{ and } Ct\,.\,Cm = CB^2.$$

Cor. 2. The axes being still supposed rectangular,

$$\therefore\ CT = \frac{a^2}{x'};\ \therefore\ MT = \frac{a^2}{x'} - x' = \frac{a^2 - x'^2}{x'},\ \text{in the ellipse,}$$

$$\text{and } = \frac{x'^2 - a^2}{x'},\ \text{in the hyperbola.}$$

this value being independent of the minor axis, it follows that for all ellipses and hyperbolas, constructed on the same major axis, the sub-tangents corresponding to the same abscissa are equal: in other words, *if the indefinite ordinate MZ meet* *the ellipses and hyperbolas described upon $A\,V$, in P, P', P'', &c., then the tangents applied at these points will all intersect the axis in the same point T.*

Observation. When any one of the ellipses described on the major axis becomes a *circle*, then if the ordinate MP to the given ellipse meet the circle in Q, tangents applied at P and Q will intersect the axis in the same point.

115. When the position of the point of contact is varied, there will be a corresponding change in the angle which the tangent makes with the axis-major. In order, therefore, to examine this change of inclination of the tangent, we shall consider what the quantity $\mp \frac{b^2}{a^2} \cdot \frac{x'}{y'}$ becomes, when different values are assigned to x'.

(1) In the ellipse (fig. 54.)

$$m = -\frac{b^2}{a^2} \cdot \frac{x'}{y'}, \text{ but } y' = \pm\frac{b}{a}\sqrt{(a^2 - x'^2)}.$$

$$\therefore m = \mp \frac{b}{a} \cdot \frac{1}{\sqrt{\frac{a^2}{x'^2} - 1}}.$$

Let $x' = 0$; in which case the point of contact is B, or b; then $m = 0$; hence the tangent at these points is parallel to $A\,V$, and therefore coincides with the tangent at the same points to the circle, described on $B\,b$.

Let $x' = \pm a$; in which case the point of contact is A or V; then m being infinite, the tangent at these points is perpendicular to $A\,V$, and therefore coincides with the tangent applied at the same points to the circle described on $A\,V$. If x' pass through all intermediate degrees of magnitude between a and 0, m will undergo a corresponding variation from 0 to infinity; therefore the tangent is inclined at all possible angles to the axis-major.

From B to A, and from b to V, it makes an obtuse angle; and from A to b, and from V to B, it makes an acute angle.

(2) In the hyperbola, (fig. 55.)

$$m = \pm \frac{b}{a} \cdot \frac{1}{\sqrt{\left(1 - \frac{a^2}{x'^2}\right)}}.$$

Let $x' = \pm a$; $\therefore$ m is infinite.

hence the tangent at A and at V is at right angles to the major axis, and therefore coincides with the tangent applied at the same points to a circle described on AV.

If the hyperbola be now referred to any system of conjugate diameters

$$m = \pm \frac{b'}{a'} \cdot \frac{1}{\sqrt{\left(1 - \frac{a'^2}{x'^2}\right)}},$$

Suppose that x' becomes infinitely great, then

$$m = \pm \frac{b'}{a'};$$

that is, the tangent to the hyperbola at a point infinitely distant, makes a finite angle with the axis. Now in general, (Art. 115.) $CT = \frac{a'^2}{x'}$, which, when x is infinite, becomes 0; therefore the tangent in this case passes through the centre.

Whence it appears, that *when the point of contact is infinitely distant, the tangent becomes an asymptote.*

116. **Prop. 7.** If at the extremities of any diameter Pp, there be drawn tangents meeting a tangent applied at any other point Q in the points T, t, then $PT . pt = CD^2$ (fig. 56.)

Let the curve be referred to the conjugate diameters Pp, Dd as axes, then, the equation to the tangent being

$$a'^2yy' + b'^2xx' = a'^2b'^2.$$

Let $x = a'$; $\therefore\ a'^2yy' = b'^2a'(a' - x')$;

$$\therefore\ y = \frac{b'^2}{a'y'}(a' - x') = PT.$$

Similarly, if $x = -a'$, $y = \frac{b'^2}{a'y'}(a' + x') = pt$;

therefore, multiplying together these values, we have

$$PT.pt = \frac{b'^4}{a'^2}.\frac{a'^2 - x'^2}{y'^2}$$
$$= b'^2 = CD^2.$$

A similar mode of proof is applicable to the hyperbola.

Cor. If the axes be rectangular, $AT.Vt = BC^2$, (fig. 57.)

Hence, in the circle, $AT.Bt = r^2$.

117. Prop. 8. If CP, CD, be a given system of semiconjugate diameters, and CP', CD' any other system, then if a tangent at P meet CP', CD' in the points T, t,

$$PT.Pt = CD^2.$$

For CP, CD being assumed as the axes, the equations to CP', CD' will be

$$y = mx, \text{ and } y = -\frac{b'^2}{a'^2m}x.$$

Now let $x = a'$; $\therefore$ y or $PT = ma'$

$$\text{and } Pt = -\frac{b'^2}{a'm};$$

therefore, multiplying these values together,

$$PT.Pt = -b'^2.$$

It may, in like manner be shewn, that in the hyperbola,

$$PT.Pt = +b'^2.$$

The reason why the sign of the product is negative in the ellipse, and positive in the hyperbola, is, that PT, Pt lie on different sides of CP in the former, and on the same side of it, in the latter.

COR. If the conjugate diameters which the tangent meets be the axes, then

$$PT \,.\, Pt = CD^2. \text{ (fig. 58.)}$$

In the circle, $PT \,.\, Pt = CP^2$, (fig. 59.)

118. PROP. 9. To draw a tangent to the ellipse and hyperbola from a given point (α, β) without the curve.

Let the curve be referred to its principal diameters; then if x', y' be the co-ordinates of the point of contact,

$$a^2yy' + b^2xx' = a^2b^2$$

will be the equation to the tangent; x', and y' being supposed unknown.

Now α, β being the co-ordinates of a point in the tangent, these quantities, when substituted for x and y, will satisfy the above equation;

$$\therefore\; a^2y' \,.\, \beta + b^2x' \,.\, \alpha = a^2b^2 \;\ldots\ldots (1).$$

But since (x', y') is a point in the curve, we have

$$a^2y'^2 + b^2x'^2 = a^2b^2 \ldots\ldots (2).$$

Hence, the quantities sought, x' and y', will be obtained, by elimination, from equations (1) and (2): and since the latter is a quadratic, there will be *two* points of contact; in other words, *two* tangents to the ellipse or hyperbola, may be drawn from a given point without the curve.

It will be easier, however, instead of actually combining these equations, to construct the loci to which they belong, since their intersection will evidently determine the points of contact. Now the locus of equation (2) is the curve itself, and since equation (1) represents a straight line, it can be no other than the line which joins the points of contact.

Whence the two tangents may thus be drawn.

Let Q (fig. 60.) be the given point (α, β), then in the equation

$$a^2y'.\beta + b^2x'.\alpha = a^2b^2.,$$

$$\text{let } y' = 0; \therefore x' = \frac{a^2}{\alpha} = CT,$$

$$x' = 0; \therefore y' = \frac{b^2}{\beta} = Ct,$$

draw Tt meeting the curve in the points $P, p,$ then, the lines QP, Qp will be the tangents required.

CHAP. IV.

ON THE PROPERTIES OF CONJUGATE DIAMETERS IN THE ELLIPSE AND HYPERBOLA.

119. **Prop. 1.** **Any two conjugate diameters of known inclination being given, to find the magnitude and direction of the principal diameters.**

Let $2a'$, $2b'$ be the conjugate diameters, γ the angle which they form, and suppose the curve referred to these diameters as axes.

It was shewn in Art. 115, that the tangent to an ellipse, at the extremity of either of the principal diameters coincides with the tangent at the same point in the circle described on that diameter. This property suggests a simple method of resolving the proposed question.

Let x', y' be the co-ordinates of the extremity of either principal diameter, and $2r$ its length.

Then the equation to the tangent at (x', y') in the ellipse is

$$a^2yy' + b^2xx' = a^2b^2 \ldots\ldots (1),$$

and that to the tangent at the same point in the circle described upon $2r$ is

$$(y' + x' \cos\gamma)\, y + (x' + y' \cos\gamma)\, x = r^2 \ldots\ldots (2),$$

and since the tangents represented by these equations coincide, we have, equating the corresponding terms,

$$\frac{y'}{b'^2} = \frac{y' + x' \cos\gamma}{r^2}, \quad \frac{x'}{a'^2} = \frac{x' + y' \cos\gamma}{r^2};$$

therefore, eliminating y',

$$\frac{b'^2 x' \cos\gamma}{r^2 - b'^2} = \frac{(r^2 - a'^2)\, x'}{a'^2 \cos\gamma};$$

hence, multiplying out and arranging the result,

$$r^4 - (a'^2 + b'^2)\, r^2 + a'^2 b'^2 \sin^2\gamma = 0 \dots\dots\dots (3),$$

which equation being resolved,

$$r^2 = \frac{a'^2 + b'^2}{2} \pm \frac{1}{2} \sqrt{\{a'^4 + b'^4 + 2a'^2 b'^2 \cos 2\gamma\}}.$$

These two values of r^2 being always real and positive, there are in the ellipse four values of r, two of which are equal, with opposite signs, to the other two. Hence, the ellipse has *two* real principal diameters.

Also, $\frac{y'}{x'} = \frac{b^2}{r^2 - b'^2} \cdot \cos\gamma$; therefore the inclination of each diameter is known.

In the hyperbola, a'^2 and b'^2 have different signs;

$$\therefore \frac{a'^2 \sim b'^2}{2} \angle \frac{1}{2} \sqrt{\{a'^4 + b'^4 - 2a'^2 b'^2 \cos 2\gamma\}},$$

whence one of the values of r^2 is negative, and two values therefore of r imaginary.

The hyperbola therefore has only *one* real principal diameter.

120. Cor. 1. Let a^2, b^2, be the roots of equation (3), then, by the theory of equations, we have

$$a^2 + b^2 = a'^2 + b'^2 \dots\dots\dots\dots (1),$$
$$ab = a'b' \sin\gamma \dots\dots\dots\dots (2).$$

From the former of these equations we infer, that "in the ellipse, the sum of the squares of any two conjugate diameters is equal to the *sum* of the squares of the principal diameters:" from the latter, that "the parallelogram, constructed upon any two conjugate diameters, is equal to the rectangle contained by the principal diameters;" in other words, that "the area of all circumscribing parallelograms, *whose sides are parallel to any two conjugate diameters*, is constant."

It may in like manner be proved, that "in the hyperbola, the *difference* of the squares of any two conjugate diameters is equal to the *difference* of the squares of the principal diameters:" and that "the area of all inscribed parallelograms, *of which the sides are parallel to any two conjugate diameters*, is constant."

121. Cor. 2. From P let a perpendicular PF be dropped upon CD, and let $PF = p$.

$$\text{Then } p = CP \sin PCD = a' \sin\gamma;$$

$$\therefore\ p \,.\, b' = a'b' \sin\gamma = ab;$$

$$p = \frac{ab}{b'} = \frac{ab}{\sqrt{(a^2+b^2-a'^2)}}, \text{ in the ellipse,}$$

$$\text{and } = \frac{ab}{\sqrt{(a'^2+b^2-a^2)}}, \text{ in the hyperbola.}$$

122. Cor. 3. It was shewn, (113.) that in the ellipse

$$\text{the normal } PG = \sqrt{\left\{a^2 - \frac{a^2-b^2}{a^2}x^2\right\}}.$$

$$\text{Now } PF = \frac{ab}{\sqrt{(a^2+b^2-a'^2)}}$$

$$= \frac{ab}{\sqrt{\left\{a^2 - \frac{a^2-b^2}{a^2}x^2\right\}}},$$

$$\text{Since } a'^2 = b^2 + \frac{a^2-b^2}{a^2}x^2 \ (97);$$

$$\therefore\ PF \,.\, PG = b^2 = BC^2;$$

a proposition which is true also when applied to the hyperbola.

123. Def. The two straight lines, which are drawn from the extremities of any diameter in the ellipse and hyperbola to any point in the curve, are called *supplemental chords*.

124. Prop. 2. If m, m', be the constant coefficients in the equations to any two supplemental chords, then $mm' = \mp \frac{b'^2}{a'^2}$ in the ellipse or hyperbola respectively.

The curve being referred to any system of conjugate diameters, let PCp be any assumed diameter, and Q any point, join PQ, pQ.

Now, if the co-ordinates of P be x', y', those of p will be $-x'$, $-y'$;

therefore, the equation to PQ is $y - y' = m(x - x')$,

and that to pQ is $y + y' = m'(x + x')$.

At the point of intersection of QP, Qp, the co-ordinates are identical; therefore, multiplying these equations together, we have

$$y^2 - y'^2 = mm'(x^2 - x'^2);$$

$$\therefore\ mm' = \frac{y^2 - y'^2}{x^2 - x'^2} = \therefore\ \mp \frac{b'^2}{a'^2}; \quad \text{Art. 99.}$$

the upper sign for the ellipse, the lower for the hyperbola.

125. Cor. 1. Let the axes be rectangular, then m, m', are the tangents of the angles which QP, Qp make with the major axis; therefore the product of these tangents $= \mp \frac{b^2}{a^2}$.

Hence, when the diameter Pp coincides with the major axis, if AQ, VQ, be drawn to any point in the ellipse,

$$\tan QAX \cdot \tan QVX = \mp \frac{b^2}{a^2}.$$

126. Cor. 2. By Art. 102. the equation to any two conjugate diameters are

$$y = mx,\ y = \mp \frac{b'^2}{a'^2} \cdot \frac{1}{m} \cdot x;$$

and since the product of the coefficients in these equations is $= \mp \frac{b'^2}{a'^2}$, it follows that *the two diameters drawn parallel to any two supplemental chords, are conjugate to one another.*

Again, each of these conjugate diameters being parallel to the tangent applied at the extremity of the other, it is evident that *the tangents themselves are parallel to the same supplemental chords.*

127. Prop. 3. An ellipse or hyperbola being traced on a plane, it is required to draw a diameter that shall be conjugate to a given diameter.

Let *Pp* be the given diameter, draw any other whatever *HCK*, through *H* draw *HQ* parallel to *Pp*, join *QK*; then, if the diameter *Dd* be drawn parallel to *QK*, it will be conjugate to *Pp*; as is evident from the last article.

128. Cor. 1. Hence is derived a simple method of drawing a tangent to an ellipse and hyperbola, from a given point in the curve.

Find (89), the centre; make the same construction as in the proposition, with this difference, that *PT*, instead of *Dd*, is to be drawn parallel to *QK*; then *PT* will obviously be the tangent required.

If the major axis of the curve be supposed given, the construction of the conjugate diameter and of the tangent will be simplified.

128. Prop. 4. To find the angle contained by the supplemental chords, drawn from the extremities of the major axis.

Let AQ, VQ be the given supplemental chords, x', y' the co-ordinates of Q, then

$$\tan Q = \tan (QAX - QVX) = \frac{\tan QAX - \tan QVX}{1 + \tan QAX \tan QVX}$$

$$= \frac{m' - m}{1 + mm'}, = \frac{m' - m}{1 \mp \frac{b^2}{a^2}}.$$

$$\text{Now } m' - m = \frac{y'}{x' - a} - \frac{y'}{x' + a} = \frac{2ay'}{x'^2 - a^2} = \frac{2ay'}{\mp \frac{a^2}{b^2} y'^2} = \mp \frac{2b^2}{ay'};$$

$$\therefore \tan Q = \frac{\mp 2b^2}{y'\left(a \mp \frac{b^2}{a}\right)} = \mp \frac{2b^2 a}{a^2 \mp b^2} \times \frac{1}{y'},$$

the upper sign for the ellipse, the lower for the hyperbola.

Hence, the angle Q is obtuse in the former, and acute in the latter.

129. Cor. 1. In the ellipse, the angle Q is the greatest possible, when y' is so; that is, when $y' = b$: the supplemental chords are, in this case, equal; and their inclination to the axis-major is

$$\tan^{-1} \pm \frac{b}{a}.$$

Hence, also, the conjugate diameters which are parallel to these chords, are equal to one another, and contain the greatest possible angle.

130. Cor. 2. In the hyperbola, the angle Q is the least, when y' is the greatest, possible; let y' be supposed infinite, then the angle Q vanishes, and the supplemental chords become parallel. Their inclination to the axis-major is easily found; for in general,

$$mm' = \frac{b^2}{a^2} = \therefore m^2, \text{ since } m = m';$$

$$\therefore m \pm \frac{b}{a},$$

the chords, therefore, are, in this case, parallel to the asymptotes.

131. Cor. 3. It appears, from a comparison of the last two corollaries, that *the asymptotes of the hyperbola are analogous to the equal conjugate diameters of the ellipse.*

Again, since the inclination of the asymptotes in the one case, and of the equal conjugate diameters in the other, $= \tan^{-1}. \frac{b}{a}$, it follows, that "if an ellipse and hyperbola be described on the same axis-major, the equal conjugate diameters of the former will be asymptotes to the latter."

132. Prop. 5. To find the magnitude and inclination of the equal conjugate diameters of the ellipse.

(1) Their magnitude.

In general, $a^2 + b^2 = a'^2 + b'^2$ (120)

$$= \therefore 2a'^2, \text{ when } a' = b';$$

$$\therefore a' = \pm \sqrt{\tfrac{1}{2}(a^2 + b^2)}.$$

(2) Their mutual inclination.

In general, $ab = a'b' \sin\gamma$

$$= \therefore a'^2 \sin\gamma, \text{ when } a' = b';$$

$$\therefore \sin\gamma = \frac{ab}{a'^2} = \frac{2ab}{a^2 + b^2}.$$

Their inclination to the axis-major is also known, since (129) it $= \tan^{-1} \pm \frac{b}{a}$.

In the common hyperbola, there cannot be two equal conjugate diameters, for since

$$a^2 - b^2 = a'^2 - b'^2,$$

if a' be supposed $= b'$, then a must $= b$, that is, the hyperbola must be *equilateral*.

133. Cor. 1. It was shewn in (129) that of all systems of conjugate diameters, those that are equal contain the greatest angle. The same may be directly proved thus;

$$\sin \gamma = \frac{ab}{a'b'};$$

$\therefore$ PCd is a minimum, or PCD a maximum, when $a'b'$ is a maximum; that is, when $a' = b'$.

134. Cor. 2. Of all systems of conjugate diameters, the sum of those that are rectangular is the least, and of those that are equal, the greatest.

$$\text{For } a' + b' = \sqrt{\{a'^2 + b'^2 + 2a'b'\}}$$

$$= \sqrt{\left\{a^2 + b^2 + \frac{2ab}{\sin \gamma}\right\}};$$

hence, $a' + b'$ is a maximum when $\sin \gamma$ is a minimum, that is, when $a' = b'$.

Also, $a' + b'$ is a minimum, when $\sin \gamma$ is a maximum, that is, when $\gamma = \frac{\pi}{2}$, or the axes are rectangular.

135. Cor. 3. The equation to the ellipse, when referred to its equal conjugate diameters, is

$$y^2 + x^2 = a'^2,$$

which is distinguished from the equation to the circle, of the same form, by the axes being oblique.

136. Prop. 6. In the ellipse, to draw two conjugate diameters that shall be inclined to one another at a given angle.

Let the curve be referred to its principal diameters, and let γ denote the *acute* angle formed by the conjugate diameters.

$$\text{Since } a' + b' = \pm \sqrt{\left\{a^2 + b^2 + \frac{2ab}{\sin \gamma}\right\}} \ (134),$$

and similarly,

$$a' - b' = \pm \sqrt{\left\{a^2 + b^2 - \frac{2ab}{\sin \gamma}\right\}},$$

we have, adding and subtracting successively these equations,

$$a' = \pm \tfrac{1}{2} \sqrt{\left\{a^2 + b^2 + \frac{2ab}{\sin \gamma}\right\}} \pm \tfrac{1}{2} \sqrt{\left\{a^2 + b^2 - \frac{2ab}{\sin \gamma}\right\}};$$

$$\text{and } b' = \pm \tfrac{1}{2} \sqrt{\left\{a^2 + b^2 + \frac{2ab}{\sin \gamma}\right\}} \mp \tfrac{1}{2} \sqrt{\left\{a^2 + b^2 - \frac{2ab}{\sin \gamma}\right\}};$$

therefore, the magnitude of the required diameters is known.

$$\text{Again, since } mm' = -\frac{b^2}{a^2}, \text{ and } m' = \frac{m - \tan \gamma}{1 + m \tan \gamma},$$

we have, by substitution and reduction,

$$m^2 - \left(1 - \frac{b^2}{a^2}\right) \tan \gamma \, . \, m = -\frac{b^2}{a^2};$$

$$\therefore m = \frac{1}{2a^2}\left[(a^2 - b^2) \tan \gamma \pm \sqrt{\{(a^2 - b^2)^2 \tan^2 \gamma - 4a^2 b^2\}}\right],$$

$$\text{these values of } m \text{ will be real, if } \tan \gamma > \frac{2ab}{a^2 - b^2}.$$

Now since γ is acute, it will be the least possible, when the diameters are equal, in which case $\tan\gamma = \frac{2ab}{a^2-b^2}$, (133). Hence, except when the diameters are equal, the quantity under the radical is positive, and the values of m, therefore, are real.

In the hyperbola, the elimination of b' will lead to an equation of the fourth degree, resolvable as a quadratic.

137. PROP. 7. In the ellipse or hyperbola, let CP, CD, be a given system of conjugate diameters, join D, P; from M any point in the curve draw Mm parallel to DP, meeting CP, CD in N, Q: it is required to prove that

$$MQ^2 \pm MN^2 = DP^2,$$

Referring the curve to CP and CD as axes,

the equation to the point D is $x=0$, $y=b'$,

and that to the point P is $y=0$, $x=a'$;

therefore the equation to DP is $y=\frac{b'}{a'}(a'-x)$;

hence, that to Mm, which is drawn parallel to DP, is

$$y-y'=\frac{b'}{a'}(x-x').$$

Suppose, now, that $x=0$, then $y-y'=-\frac{b'}{a'}x'$,

and $x-x'=-x'$;

$$\therefore\ MN^2=x'^2+\frac{b'^2}{a'^2}x'^2+2\frac{b'}{a'}x'^2\cos x,y\ (55)$$

$$=\frac{x'^2}{a'^2}\{a'^2+b'^2+2a'b'\cos x,y\}\ldots\ldots\ldots(1).$$

Similarly,

$$MQ^2 = \frac{y'^2}{b'^2}\{a'^2 + b'^2 + 2a'b' \cos. x, y\} \ldots\ldots\ldots (2);$$

$$\therefore\ MQ^2 \pm MN^2 = \left(\frac{y^2}{b'^2} \pm \frac{x^2}{a'^2}\right)\{a'^2 + b'^2 + 2a'b' \cos. x, y\}$$

$$= a'^2 + b'^2 + 2a'b' \cos x, y \quad (99)$$

$$= DP^2.$$

Hence, $MQ^2 \pm MN^2$ is a constant quantity.

◆

CHAP. V.

ON THE FOCI AND POLAR EQUATIONS OF THE ELLIPSE AND HYPERBOLA.

138. THE equation to the ellipse, in terms of its principal parameter is

$$y^2 = 2px - \frac{p}{a}x^2,$$

which, when $p = a$, becomes the equation to the circle. It appears, then, that in the circle, *the ordinate drawn through the centre, is equal to half the principal parameter.*

From the analogy subsisting among lines of the second order, we are thence naturally led to inquire whether the same property can be extended to the ellipse and hyperbola.

139. PROP. 1. To find whether there be any point in the major axis of the ellipse and hyperbola, at which the ordinate is equal to the principal semi-parameter.

The centre being supposed the origin, let x be the abscissa of the point sought;

Then in general $y^2 = \pm \frac{b^2}{a^2}(a^2 - x^2)$,

and $= \frac{b^4}{a^2}$, by hypothesis.

therefore equating these values of y^2, and dividing each side by $\frac{b^2}{a^2}$, we have $\pm b^2 = a^2 - x^2$;

$$\therefore x = \pm \sqrt{(a^2 \mp b^2)};$$

there are *two* points, therefore, in the major axis, at which the property above enunciated obtains; to determine them, from C, as a centre, with radius $= \sqrt{(a^2 - b^2)}$ in the ellipse, and $= \sqrt{(a^2 + b^2)}$ in the hyperbola, describe a circle cutting AV in the points S, H: then, S and H are the points required. Each of them is called the *focus*, and the distance of either from any point in the curve is called the *focal distance.*

140. Cor. 1. The distance of either focus from the centre, is called the *eccentricity*, and is commonly denoted by ae; (e) being < 1 in the ellipse, and > 1 in the hyperbola.

141. Cor. 2. Since $1 - e^2 = \pm \frac{b^2}{a^2}$, the equations to the ellipse and hyperbola become, by substitution,

$$y^2 = (1 - e^2)(a^2 - x^2).$$

142. Prop. 2. To find the distance of any point in the ellipse or hyperbola, from either focus.

Let S, H be the foci, P any point in the curve, and r the distance sought.

Then, (1) in the ellipse, $r^2 = (x - x')^2 + (y - y')^2$.

But $y' = 0$, since the foci are situated on the axis of x;

$$\therefore r^2 = (x - x')^2 + y^2$$
$$= x^2 - 2xx' + x'^2 + (1 - e^2)(a^2 - x^2).$$

Now if we confine our attention to the focus S, then

$$x' = + ae;$$

therefore, substituting,

$$r^2 = x^2 - 2aex + a^2e^2 + (1-e^2)(a^2 - x^2)$$
$$= a^2 - 2aex + e^2x^2;$$
$$\therefore\ SP = \pm (a - ex).$$

Since r is not measured *along a line given in position*, the rule for the interpretation of the signs is, in this case, inapplicable. The negative value of r being, therefore, merely symbolical, is to be rejected; Schol. p. 35.

Hence, $SP = a - ex$; and similarly, $HP = a + ex$.

(2) In like manner it may be proved, that in the hyperbola,

$SP = ex - a$; and $HP = ex + a$.

The distance, therefore, of either focus from any point in the curve, is a rational function of the corresponding abscissa.

143. Cor. In the ellipse, $SP + HP = 2a$,

and in the hyperbola, $HP - SP = 2a$,

that is, in the ellipse, the *sum*, and in the hyperbola, the *difference*, of the focal distances, is equal to the major axis.

144. *Observation.* If the curve, instead of being referred to its principal diameters, be referred to *any system of conjugate diameters*, it may be proved, precisely as above, that in the axis of x, there are always two points, such that the corresponding ordinate is equal to the semi-parameter: the abscissa being in that case $x = \pm \sqrt{(a'^2 \mp b'^2)}$.

It thence follows, that if $2a'$, $2b'$, be any two conjugate diameters, there are two points in the former analogous to the foci; they are distinguished from the *foci*, properly so called, by this circumstance, that their distance from any point in the curve, is not a rational function of the abscissa.

145. Prop. 3. To find the polar equations to the ellipse and hyperbola, either focus being the pole.

(1) To the ellipse.

Let S be the focus, $\angle ASP = \omega$, $SP = r$ (fig. 45.)

The equation to the ellipse, when referred to rectangular axes, is (141),

$$y^2 = (1 - e^2)(a^2 - x^2) \ldots\ldots (1),$$

hence, in order to obtain the polar equation, it is only necessary to substitute $r \sin \omega$ for y, and $ae + r \cos \omega$ for x;

Equation (1) when thus transformed becomes

$$r^2 \sin^2 \omega = (1 - e^2)\left\{a^2 - (ae + r \cos \omega)^2\right\},$$

whence, we have

$$r^2 + \frac{2(1 - e^2)\,ae \cos \omega}{1 - e^2 \cos^2 \omega}\,r = \frac{a^2(1 - e^2)^2}{1 - e^2 \cos^2 \omega};$$

$$\therefore\ r = \frac{1}{1 - e^2 \cos^2 \omega}\left\{-(1 - e^2)\,ae \cos \omega \pm a(1 - e^2)\right\}$$

$$= a(1 - e^2)\,\frac{\pm 1 - e \cos \omega}{1 - e^2 \cos^2 \omega}$$

Let the positive and negative sign be successively taken, and let the resulting values of r be denoted by ρ, ρ', respectively; then, since

$$1 - e^2 \cos^2 \omega = (1 + e \cos \omega)(1 - e \cos \omega),$$

we have

$$\rho = a\,\frac{1 - e^2}{1 + e \cos \omega} \ldots\ldots\ldots\ldots (1),$$

$$\rho' = -a\,\frac{1 - e^2}{1 - e \cos \omega} \ldots\ldots\ldots (2).$$

(2) To the hyperbola.

Let S be the focus, $\angle ASP = \omega$, and $SP = r$ (fig. 46).

Then it may be proved, as in the case of the ellipse, that if ρ, ρ' denote the two values of r,

$$\rho = a\,\frac{e^2 - 1}{1 + e \cos \omega} \ldots\ldots\ldots\ldots (1),$$

$$\rho' = -a\,\frac{e^2 - 1}{1 - e \cos \omega} \ldots\ldots\ldots (2).$$

Cor. When the focus H is assumed as the pole, the equations are in each case similar to those already obtained.

146. **Prop. 4.** To trace the ellipse and hyperbola, by means of their polar equations.

(1) To trace the ellipse;

$$\text{Here } \rho = a\,\frac{1-e^2}{1+e\cos\omega}, \text{ and } \rho' = -a\,\frac{1-e^2}{1-e\cos\omega}.$$

Now (e) being $\angle 1$, the numerator $1-e^2$ is positive, and since $\cos\omega$ cannot exceed the radius, the denominator is also positive. Whence, of these two values of r, the first is essentially positive, and the second essentially negative; therefore, consistently with the principles stated in the Schol. p. 35, and in Art. 80, the value of ρ' must be considered *imaginary*. It is only necessary, therefore, to discuss the first value of r.

$$\text{Now let } \omega = 0; \;\therefore\; \rho = a\,\frac{1-e^2}{1+e} = a(1-e) = SA,$$

$$\omega = \frac{\pi}{2}; \;\therefore\; \rho = a(1-e^2) = \frac{b^2}{a} = SL,$$

$$\omega = \pi; \;\therefore\; \rho = a\,\frac{1-e^2}{1-e} = a(1+e) = SV,$$

$$\omega = \frac{3\pi}{2}; \;\therefore\; \rho = a(1-e^2) = \frac{b^2}{a} = Sl,$$

$$\omega = 2\pi; \;\therefore\; \rho = a(1-e) = SA.$$

The different points of the ellipse may, therefore, be determined by supposing the angle ω to vary from 0 to 360^0; the radius vector r being, in all cases, positive.

(2) To trace the hyperbola;

$$\text{Here } \rho = a\,\frac{e^2-1}{1+e\cos\omega}, \text{ and } \rho' = a\,\frac{e^2-1}{e\cos\omega-1}.$$

Since $e > 1$, the numerator of each is positive, and the signs of ρ and ρ' will depend on the sign of the denominator. Now if ω be acute, $\cos\omega$ is positive, therefore, ρ is always real, and ρ' will be real or imaginary, according as $e\cos\omega - 1$ is $>$ or < 0. Again, if ω be obtuse, $\cos\omega$ is negative, whence ρ' is always imaginary, and ρ will be real or imaginary, according as $1 - e\cos\omega > \text{or} < 0$.

Consequently, of these two values of r, neither is essentially positive nor essentially negative: in order, therefore, to determine the curve, we must discuss each of them separately.

First, to discuss ρ.

Let $\omega = 0$; $\therefore\ \rho = a\dfrac{e^2-1}{e-1} = a(e-1) = SA$,

$\omega = \dfrac{\pi}{2}$; $\therefore\ \rho = a\dfrac{e^2-1}{1} = \dfrac{b^2}{a} = SL$,

$\omega > \dfrac{\pi}{2}$; $\therefore$ $\cos\omega$ being negative, ρ will increase, and it will continue to do so, until $1-e\cos\omega=0$, when it will become infinite. The value of ω is then

$$= \cos^{-1}\frac{1}{e} = \sec^{-1} e = \tan^{-1}\sqrt{(e^2-1)} = \tan^{-1} \pm \frac{b}{a},$$

in which, since ω is obtuse, the lower sign must be taken.

Now $\tan^{-1} - \dfrac{b}{a}$ is the inclination of the asymptote to the axis (103); therefore, when ρ is parallel to CR it becomes infinite.

Let $\omega = \pi$; $\therefore$ ρ being negative, is *imaginary*.

Let ω be now reckoned *below* AX; then, since the cosine of an arc is the same whether the arc be positive or negative, the values of ρ in the branch Az may be found precisely as before. Wherefore, ρ determines exclusively the *right* hand branch of the hyperbola, the angle ω being supposed to vary from 0 to $\tan^{-1} - \dfrac{b}{a}$.

Next, to discuss ρ'.

Let $\omega=0$; $\therefore\ \rho' = a\dfrac{e^2-1}{e-1} = a(e+1) = SV$,

$\omega < \dfrac{\pi}{2}$; $\therefore$ $\cos\omega$ being positive, ρ' increases; and it will continue to do so until $e\cos\omega - 1 = 0$, when it becomes infinite.

In this case, $\omega = \cos^{-1}\frac{1}{e} = \tan \pm \frac{b}{a}$, in which the upper sign must be used, since ω is acute.

But $\tan^{-1} + \frac{b}{a}$ is the inclination of the asymptote CR' to the axis; therefore, when ρ' is parallel to CR', it becomes infinite.

Let $\omega = \frac{\pi}{2}$; $\therefore$ ρ' being negative, ω is imaginary.

In like manner, the lower branch Vz may be traced.

Wherefore, ρ' belongs exclusively to the *left* hand branch of the hyperbola, ω being made to vary from 0 to $\tan^{-1} + \frac{b}{a}$.

It appears from this discussion, that of the two values of r, one is appropriated to the right hand branch of the hyperbola, and the other to the left; in the ellipse, on the contrary, one value of r is symbolical, and the curve may be completely traced by means of the other.

147. PROP. 5. To find the locus of a point, the *sum* or *difference* of whose distances from two fixt points is given.

Let S, H be the two fixt points, P the point whose locus is sought.

Bisect SH in C, drop the perpendicular PM on SH, produce HS to A, and assume

$$SP = r,\quad HP = r',\quad \angle\, PSA = \omega,$$
$$CM = x,\quad MP = y,\quad \text{and}\quad SC = c.$$

(1) Suppose the *sum* to be given, and to $= 2a$.

$$\text{Then } r^2 = y^2 + (c - x)^2 \ldots\ldots (1),$$
$$\text{and } r'^2 = y^2 + (c + x)^2 \ldots\ldots (2);$$
$$\therefore r'^2 - r^2 = (c + x)^2 - (c - x)^2 = 4cx,$$
$$\text{but } r'^2 - r^2 = (r' + r)(r' - r) = 2a(r' - r);$$
$$\therefore 2a(r' - r) = 4cx; \quad \therefore r' - r = \frac{2cx}{a}.$$

$$\text{Now } r'+r=2a;\ \therefore r'=a+\frac{cx}{a}\ \ldots\ldots (3),$$

$$\text{and } r=a-\frac{cx}{a}\ \ldots\ldots (4).$$

Taking the second of these equations, and substituting in it for x its equivalent $c+r\cos\omega$, we have

$$r=a-\frac{c}{a}(c+r\cos\omega);$$

$$\therefore r=\frac{a^2-c^2}{a+c\cos\omega},$$

which (145) is the polar equation to an ellipse. **The locus,** therefore, of P is an ellipse whose *foci* are S, H, and whose major axis is $=2a$.

(2) If the difference be given, it may, in like manner, be shewn, that

$$r=\frac{c^2-a^2}{a+c\cos\omega}.$$

which is the polar equation to an hyperbola.

The locus, therefore, in this case is an hyperbola, whose foci are S, H, and whose major axis $=2a$.

148. Cor. If it were required to deduce the *algebraic* equations to the point P, we proceed as follows:

(1) Let the sum of the distances be given.

Squaring each of the equations (3) and (4), and adding the result, we have

$$r^2+r'^2=2\left(a^2+\frac{c^2}{a^2}x^2\right),$$

$$\text{but } r^2+r'^2=2(y^2+c^2+x^2),$$

from equations (1) and (2); therefore equating these values,

$$a^2+\frac{c^2}{a^2}x^2=y^2+c^2+x^2;$$

$$\therefore y^2 = a^2 - c^2 + \frac{c^2}{a^2}x^2 - x^2$$

$$= \frac{a^2 - c^2}{a^2}(a^2 - x^2).$$

Let $x = 0$; $\therefore y^2 = a^2 - c^2 = b^2$, if b = the ordinate drawn from C;

$$\therefore y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

which is the equation to an ellipse whose axes are $2a$ and $2b$, and whose centre is C.

(2) In the same manner, when the difference of the distances is given, the equation to the point P is

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2),$$

which represents an hyperbola, whose axes are $2a$, $2b$, and whose centre is C.

149. PROP. 7. To find the polar equation to the ellipse and hyperbola, the *centre* being the pole.

(1) To the ellipse.

Let $CP = \rho$, $\angle PCA = \theta$,

$$\begin{aligned} \text{then } \rho^2 &= x^2 + y^2 = x^2 + (1 - e^2)(a^2 - x^2) \\ &= e^2x^2 + a^2(1 - e^2) \\ &= e^2\rho^2\cos^2\theta + a^2(1 - e^2); \end{aligned}$$

$$\therefore \rho^2(1 - e^2\cos^2\theta) = a^2(1 - e^2);$$

$$\therefore \rho = a\sqrt{\frac{1 - e^2}{1 - e^2\cos^2\theta}}.$$

(2) To the hyperbola.

Since $e^2 > 1$, we have, in this case,

$$\rho = a\sqrt{\frac{e^2 - 1}{e^2\cos^2\theta - 1}}.$$

In each curve ρ is to be considered positive, and the negative value of ρ obtained by extracting the square root is to be rejected as imaginary.

150. **Cor.** In the ellipse,

$$\text{Since } b = a\sqrt{(1-e^2)};$$

$$\therefore \rho = b\,(1 - e^2\cos^2\theta)^{-\frac{1}{2}}.$$

Now when the eccentricity is very small, we may, without any sensible error, neglect the powers of (e) above the second; therefore, developing, we have

$$\rho = b\,(1 + \tfrac{1}{2}\,e^2\cos^2\theta), \text{ nearly};$$

$$\therefore \rho - b = \tfrac{1}{2}\,be^2\cos^2\theta \;\ldots\ldots\ldots\; (1).$$

$$\text{Similarly, } a - \rho = \tfrac{1}{2}ae^2\sin^2\theta \;\ldots\ldots\ldots\; (2).$$

Hence, "in ellipses of small eccentricity, the *increment* of the radius vector, in moving from B towards A, is very nearly proportional to $\cos^2 PCA$; and its *decrement*, in moving from A towards B, is very nearly proportional to $\sin^2 PCA$."

151. **Prop.** 8. The rectangle contained by the focal distances of any point, is equal to the square of the semi-diameter conjugate to that which passes through the given point.

(1) In the ellipse,

$$\begin{aligned} CD^2 &= a^2 + b^2 - CP^2 \\ &= a^2 + b^2 - (x^2 + y^2) \\ &= a^2 - \frac{a^2 - b^2}{a^2}\,x^2 = a^2 - e^2x^2 \;\ldots\; (1). \end{aligned}$$

$$\begin{aligned} \text{Now } a^2 - e^2x^2 &= (a + ex)(a - ex) \\ &= \therefore HP\,.\,SP\,.\,.\,(2); \end{aligned}$$

therefore, equating (1) and (2),

$$SP\,.\,PH = CD^2.$$

(2) The method of proof is precisely similar in the hyperbola.

152. **Prop. 9.** In the ellipse and hyperbola, the normal at any point bisects the angle formed by the focal distances.

$$\text{For } CG = CM - MG = x\left(1 \mp \frac{b^2}{a^2}\right) = \therefore e^2x;$$

$$\therefore \frac{SG}{HG} = \frac{SC - CG}{SC + CG} = \frac{ae - e^2x}{ae + e^2x} = \frac{a - ex}{a + ex} = \therefore \frac{SP}{HP};$$

therefore, (*Eucl.* vi. 3.) the ∠ SPH is bisected by PG.

153. **Cor.** Hence, the focal distances make equal angles with the tangent; a property which suggests an obvious method of drawing a tangent geometrically at a given point in the given curve.

154. **Prop. 10.** If perpendiculars be dropped from the foci upon the tangent at any point, the rectangle contained by them is equal to the square of the semi-axis minor.

Let the perpendiculars SY, HZ be dropped from the foci on the tangent at any point $P(x', y')$,

$$\text{then } SY \cdot HZ = BC^2, \text{ (fig. 60.)}$$

$$\text{For } SY = ST \cdot \sin T = (CT - CS) \sin T$$

$$= \left(\frac{a^2}{x'} - ae\right) \sin T.$$

$$\text{Similarly, } HZ = \left(\frac{a^2}{x'} + ae\right) \sin T;$$

$$\therefore SY \cdot HZ = \left(\frac{a^4}{x'^2} - a^2e^2\right) \sin^2 T = \frac{a^2}{x'^2}(a^2 - e^2x'^2) \sin^2 T \; (1)$$

$$\text{Now } \tan T = -\frac{b^2}{a^2} \cdot \frac{x'}{y'}, \text{ (108)};$$

$$\therefore \sin^2 T = \frac{\tan^2 T}{1 + \tan^2 T} = \frac{b^2}{a^2} \cdot \frac{x'^2}{a^2 - e^2x'^2};$$

therefore, by substitution in (1), we have

$$SY . HZ = b^2 = BC^2.$$

The proposition is proved in precisely the same manner in the hyperbola.

155. Cor. Since the focal distances make equal angles with the tangent, the triangles SPY, HPZ are similar,

$$\therefore\ HZ = SY . \frac{HP}{SP};$$

$$\therefore\ SY . HZ = SY^2 . \frac{HP}{SP} = BC^2;$$

$$\therefore\ SY^2 = BC^2 . \frac{SP}{HP}.$$

Let $SY = p$, $SP = r$, then $p^2 = b^2 . \dfrac{r}{2a \mp r}$ in the ellipse or hyperbola, respectively.

156. Prop. 11. To find the locus of the points Y and Z, (fig. 60.)

From T drop the perpendicular TQ on DC produced, and let YS meet CQ in the point q. Then

$$\begin{aligned} CY^2 = Cq^2 + qY^2 = Cq^2 + TQ^2 & \\ &= a^2e^2 \cos^2 T + \frac{a^4}{x'^2} \sin^2 T \\ &= a^2e^2 - \left(a^2e^2 - \frac{a^4}{x'^2}\right) \sin^2 T \\ &= a^2e^2 - \frac{a^2}{x'^2} (e^2x'^2 - a^2) \sin^2 T \ldots\ldots (1), \end{aligned}$$

but $\sin^2 T = \dfrac{b^2}{a^2} . \dfrac{x'^2}{a^2 - e^2x'^2} = (1 - e^2) \dfrac{x'^2}{a^2 - e^2x'^2}$;

therefore substituting this value in (1), we have

$$CY^2 = a^2e^2 + a^2(1 - e^2)$$

$$= a^2;$$

$$\therefore CY = a.$$

In like manner, $CZ = a$.

Hence the locus of the points Y, Z is a circle whose diameter is the major axis.

CHAP. VI.

ON THE PARABOLA.

157. SINCE the parabola is only a species of the ellipse or hyperbola, its equation may be immediately deduced from that to either of the latter curves, by supposing the centre removed to an infinite distance. The different properties of the parabola might, in like manner, be inferred from the analogous ones of the ellipse and hyperbola; but it will be more satisfactory to derive them at once from its equation.

158. PROP. 1. To find the equation to the parabola, when referred to rectangular axes, (fig. 61.)

The equation to the ellipse and hyperbola, in terms of their principal parameter is, (101)

$$y^2 = 2px \mp \frac{p}{a}x^2 \; \ldots\ldots \; (1).$$

$$\text{Here } p = \frac{b^2}{a} = \frac{a^2 - a^2e^2}{a} = \frac{(a+ae)(a-ae)}{a},$$

$$\text{Now } a - ae = AS,$$

$$\text{and } a + ae = 2a,$$

when the centre is infinitely distant;

$$\therefore p = \frac{2AS \,.\, a}{a} = 2AS,$$

$$\text{and } \frac{p}{a} = \frac{2AS}{a},$$

which is a quantity indefinitely small; therefore equation (1) becomes, by substitution,

$$y^2 = 4AS \, . \, x = 2px,$$

if p be assumed $= 2AS$.

The point S is called the focus, A the principal vertex, and the line AX the axis, of the parabola. The principal parameter is sometimes called the *latus rectum*.

159. Cor. There is only one focus in the axis of the parabola.

For the focus being that point at which the ordinate is equal to the semi-parameter,

$$y^2 = p^2, \text{ and also} = 2px;$$

$$\therefore x = \frac{p}{2} = AS.$$

160. Prop. 2. To trace the figure of the parabola by means of its equation.

$$\text{Since } y = \pm \sqrt{2px}.$$

For each value of x there are two equal values of y with opposite signs.

$$\text{Let } x = 0; \; \therefore y = 0.$$

Hence the curve passes through the origin.

In proportion as x increases, y increases, and when x becomes infinitely great, y likewise becomes so.

Let x be negative, then y is imaginary, and no point of the parabola, is situated to the left of A.

The parabola has, therefore, the form assigned to it in the figure already referred to.

161. **Prop. 3. A parabola being traced upon a plane, to find the position of its axis.**

Let PQ, pq be any two parallel chords, and through their middle points R, r draw rRZ, which will be a diameter; from any point H of the parabola, drop HK perpendicular to rZ, and produce it to meet the curve in K; bisect HK in L; then ALX being drawn parallel to rZ, will be the axis required; as is quite evident.

162. **Prop. 4. To find the distance of any point from the focus.**

Let (r) be the distance sought; then

$$r^2 = SM^2 + MP^2 = \left(x - \frac{p}{2}\right)^2 + y^2$$

$$= \left(x - \frac{p}{2}\right)^2 + 2px$$

$$= x^2 - px + \frac{p^2}{4} + 2px$$

$$= \left(x + \frac{p}{2}\right)^2;$$

$$\therefore r = \pm\left(x + \frac{p}{2}\right)$$

the positive sign being alone significant.

Hence, if in SA produced, AB be taken $= AS = \frac{p}{2}$, and BQ be drawn at right angles to AX, we shall have

$$SP = AM + AB = BM = PQ,$$

in other words, *the focal distance* $PS =$ *the perpendicular distance from a fixt line* BZ. The parabola may, therefore, be considered as the locus of a point whose distances from a fixt point, and from a line given in position, are always equal. The line BZ is called the *directrix*.

163. **Cor.** The polar equation to the parabola is deducible immediately from the foregoing proposition.

Let $ASP' = \omega$.

$$\text{Then } r = \frac{p}{2} + x$$

$$= p - r\cos\omega;$$

$$\therefore r = \frac{p}{1+\cos\omega}, \quad \text{or} = \frac{p}{2\cos^2\frac{\omega}{2}}.$$

164. **Prop. 5.** To find the equation to the parabola, when referred to any diameter and the tangent at its extremity as axes, the origin being on the curve.

The corresponding equation to the ellipse and hyperbola is (101)

$$y^2 = 2px \mp \frac{p}{a'}x^2 \ldots\ldots\ldots (1).$$

$$\text{Here } p = \frac{b'^2}{a'} = \frac{SP.PH}{a'}(151) = \frac{r}{a'}(2a - r) = 2r\,.\,\frac{a}{a'} - \frac{r^2}{a'}.$$

Now when the centre is infinitely distant, $a = a'$, because, in general,

$$CP \text{ or } a' = a\sqrt{\frac{1-e^2}{1-e^2\cos^2\theta}}.$$

and, in the case supposed, $\theta = 0$; $\therefore \cos^2\theta = 1$, and $a' = a$;

$$\therefore p = 2r - \frac{r^2}{a'} = 2r, \text{ since } \frac{r^2}{a'} \text{ vanishes;}$$

therefore, substituting in (1),

$$y^2 = 4rx \mp \frac{2r}{a'}x^2$$

$$= \therefore 4rx,$$

which is the equation required.

The line $4SP$, which is the value of the parameter in the ellipse or hyperbola, when either passes into a parabola, is called the *parameter of the diameter passing through* P.

Let $2p=4SP$.

Then the equation to the parabola becomes

$$y^2=2px,$$

which is of the same form with that in (158).

The axes to which the parabola is referred, are, by analogy, called *conjugate axes*.

165. Cor. 1. Let vQ be any ordinate to the diameter, passing through P.

Then, we have, by the equation,

$$4SP\,.\,Pv=Qv^2.$$

166. Cor. 2. Lines of the second order, when referred to any diameter, and the tangent at its extremity, may be defined by the equation,

$$y^2=mx+nx^2,$$

(n) being *negative* in the ellipse, *positive* in the hyperbola, and equal to nothing in the parabola.

167. Prop. 6. To find the equation to a tangent, at any point (x', y') of a parabola, when referred to any system of conjugate diameters.

The equation to a secant passing through that point is

$$y-y'=m(x-x')\,\ldots\ldots\ldots(1).$$

Now, $y^2=2px$, and $y'^2=2px'$;

therefore, subtracting, $y^2-y'^2=2p(x-x')$,

$$\text{or } (y+y')(y-y')=2p(x-x'),$$

and substituting for $y-y'$, its value in (1)

$$(y+y').m(x-x')=2p(x-x'),$$

$$\text{or } m=\frac{2p}{y+y'}.$$

Let now $y=y'$, then the secant becomes a tangent;

$$\text{and } m=\frac{p}{y'};$$

therefore, the equation to the tangent is

$$y-y'=\frac{p}{y'}(x-x')\ldots\ldots\ldots(2),$$

or, by reduction,

$$yy'=p(x+x')\ldots\ldots\ldots(3).$$

168. Cor. 1. When the axes are rectangular, the equation to the tangent becomes

$$yy'=p(x+x'),\ p \text{ being } = 2AS;$$

and therefore, the equation to the normal is

$$yy'=-\frac{1}{p}(x+x'),\ \text{or } y-y'=-\frac{y'}{p}(x-x').$$

169. Cor. 2. Hence may be found the magnitude of the sub-tangent, and of the sub-normal.

(1) Of the sub-tangent.

In the equation to the tangent, let $y=0$;

$$\therefore\ x+x'=0,\ \text{or } x=-x',$$

$$\text{that is, } AM=AT,$$

or $MT=MA$, that is, the *sub-tangent is equal to twice the abscissa*.

(2) Of the sub-normal.

The equation to the normal being

$$y-y'=-\frac{y'}{p}(x-x'),$$

$$\text{if } y=0, \text{ then } x-x'=MG;$$

therefore, dividing each side by y', $-1=-\frac{1}{p}.MG$;

$$\therefore\ MG=p=\tfrac{1}{2}\text{ latus rectum.}$$

170. A tangent may be drawn to a parabola from a point without it, on the same principle with that employed in Art. 118.

For, let x', y' be the co-ordinates of the given point,

x, y those of the point of contact.

Then, the equation to the tangent is

$$yy'=p(x+x')\ldots\ldots\ldots(1),$$

$$\text{also, } y^2=2px\ldots\ldots\ldots\ldots(2).$$

From these two equations x and y may be obtained by elimination.

But, instead of eliminating, it is more convenient to construct the loci of (1) and (2). Now, the former is the equation to the line joining the points of contact, and the latter, the equation to the parabola. Hence, the two tangents may be drawn.

171. PROP. 7. If from the point of contact two straight lines be drawn, one to the focus and the other parallel to the axis, they will form equal angles with the tangent.

$$\text{For } ST=SA+AT=\frac{p}{2}+x,$$

$$\text{and } SP \quad \text{also} \quad =\frac{p}{2}+x;$$

$$\therefore \text{ angle } SPT=\text{angle } STP=\text{angle } tPQ.$$

172. COR. This property leads to a simple method of drawing geometrically a tangent to a parabola.

Let the point be in the parabola, as at P, fig. 61.

From P draw PS to the focus, and drop the perpendicular MQ on the directrix, join QS: then, if PT be drawn perpendicularly to SQ, it will be a tangent at P, as is evident.

If the point be without the parabola, the construction is a little different: but it would be superfluous to dwell upon it, as the analytical method above laid down is preferable.

◆

CHAP. VII.

ON THE ASYMPTOTES OF THE HYPERBOLA.

173. **Prop. 1.** **The hyperbola is the only line of the second order, that admits of asymptotes.**

The general equation to lines of the second order, when the axes are supposed to be a system of conjugate diameters, originating at the principal vertex, is

$$y^2 = mx + nx^2;$$

therefore, extracting the square root,

$$y = \pm n^{\frac{1}{2}} x \left\{1 + \frac{m}{n} \cdot \frac{1}{x}\right\}^{\frac{1}{2}}$$

$$= \pm n^{\frac{1}{2}} x \left\{1 + \frac{1}{2} \cdot \frac{m}{n} \cdot \frac{1}{x} - \frac{1}{8} \cdot \frac{m^2}{n^2} \cdot \frac{1}{x^2} \ldots\right\}$$

$$= \pm \left\{n^{\frac{1}{2}} x + \frac{1}{2} \cdot \frac{m}{n^{\frac{1}{2}}} - \frac{1}{8} \cdot \frac{m^2}{n^{\frac{3}{2}}} \cdot \frac{1}{x} \ldots\ldots\right\},$$

which is of the same form with that of the equation in Art. 105; therefore, x being assumed indefinitely great, the equation to the rectilinear asymptotes is

$$y = \pm \left(n^{\frac{1}{2}} x + \frac{1}{2} \cdot \frac{m}{n^{\frac{1}{2}}}\right).$$

Now, in the parabola $n = 0$; in the ellipse n is negative, and therefore $n^{\frac{1}{2}}$ is imaginary, and in the hyperbola n is positive: hence, the hyperbola alone admits of asymptotes, as was to be proved.

174. Cor. Comparing the equation $y^2 = mx + nx^2$, with that to the hyperbola,

$$y^2 = \frac{b'^2}{a'^2}(2a'x + x^2),$$

$$\text{we have } m = \frac{2b'^2}{a'}, \text{ and } n = \frac{b'^2}{a'^2};$$

therefore, by substitution, the equation to the asymptotes becomes

$$y = \pm\left(\frac{b'}{a'}x + b'\right).$$

At the point where the asymptotes intersect one another,

$$\frac{b'}{a'}x + b' = 0, \text{ or } x = -a',$$

which is the abscissa of the centre. Hence, the asymptotes meet at the centre, which agrees with what has been already stated.

Cor. When the axes are rectangular, and originate at the centre, the equation to the asymptotes is

$$y = \pm\frac{b}{a}x.$$

This equation becomes, in the equilateral hyperbola,

$$y = \pm x,$$

therefore, the asymptotes are at right angles to one another, and each is inclined to the axis at an angle of 45^0.

175. Prop. 2. If a straight line be drawn through any point in the hyperbola, and be produced to meet the asymptotes, the parts intercepted between the curve and the asymptotes are equal (fig. 62.).

Let *RPpr* be a line drawn through any point *P*, and produced both ways to meet the asymptotes in *R* and *r*: to prove that, $PR = pr$.

Let the hyperbola and its asymptotes be referred to the same system of conjugate diameters CX, CY, of which CX is parallel to Rr: then CX bisects the ordinate to the asymptotes as well as that to the curve; that is, $MR=Mr$, and $MP=Mp$; $\therefore\ PR=pr$.

176. Cor. 1. Let Rr move parallel to itself till the two points P, p coalesce, then, Rr becomes a tangent to the hyperbola, and therefore, by the property above proved, is bisected at the point of contact.

Cor. 2. Hence, $PR\,.\,Pr=CD^2$.

For $PR\,.\,Pr=(MR-MP)\,(MR+MP)$

$$=MR^2-MP^2=\frac{b'^2}{a'^2}x^2-\frac{b'^2}{a'^2}(x^2-a'^2)=b'^2=CD^2.$$

177. Prop. 3. To find the equation to the hyperbola, when referred to its asymptotes (fig. 69.).

Let P be any point in the hyperbola, PM, MC its rectangular co-ordinates, PR, RC its co-ordinates referred to the asymptotes CZ, Cz.

Through R and P draw Rm, PQ parallel to PM, CX respectively.

$$\text{Let }\left.\begin{matrix} CM=x, & MP=y \\ CR=x', & RP=y' \end{matrix}\right\},\ \text{and angle } RCX=\theta.$$

The aim of the problem is to express x' and y' in terms of x and y, since the equation sought will then be obtained by substituting these values of x' and y', in the equation

$$a^2y^2-b^2x^2=-a^2b^2.$$

Now $y=PM=Rm-RQ=CR\sin RCX-PR\sin RCX$;

$$\therefore\ y=(x'-y')\sin\theta,$$

$$=(x'-y')\frac{b}{\sqrt{(a^2+b^2)}},\ \text{since } \tan\theta=\pm\frac{b}{a}.$$

Similarly, $x=(x'+y')\dfrac{a}{\sqrt{(a^2+b^2)}}$;

$$\therefore\ a^2y^2-b^2x^2=-4x'y'\cdot\frac{a^2b^2}{a^2+b^2}=\text{also }-a^2b^2;$$

$$\therefore\ x'y'=\tfrac{1}{4}(a^2+b^2),$$

which is the equation required.

Hence, suppressing the accents, and denoting $\frac{1}{4}(a^2+b^2)$ by m^2, the equation to the hyperbola referred to its asymptotes, is $xy=m^2$.

178. Cor. 1. If Pp be drawn parallel to CR, then, the area of the parallelogram Rp is constant.

$$\begin{aligned}\text{For its area }&=RC\,.\,Cp\,.\sin RCz\\&=xy\sin x,y\\&=m^2\sin x,y,\end{aligned}$$

whence the truth of the corollary.

179. Cor. 2. Let the axes be parallel to the asymptotes, and originate at a point (a,β).

$$\begin{aligned}\text{Then }x&=a+x',\\y&=\beta+y'\end{aligned}$$

Substituting these values of x and y in the equation $xy=m^2$, we have

$$x'y'+ay'+\beta x'+a\beta-m^2=0.$$

Comparing this with the equation

$$bxy+dy+ex+f=0,$$

it appears that the general equation, when deprived of the terms involving y^2 and x^2, represents an hyperbola, referred to axes which are parallel to the asymptotes.

180. Prop. 4. To find the equation to the tangent, the asymptotes being the axes.

The equation to a secant drawn through any two points (x', y') and (x'', y'') is

$$y-y' = \frac{y''-y'}{x''-x'}(x-x') \ldots\ldots (1).$$

Now these points being in the hyperbola,

$$x'y' = m^2 = x''y'';$$

$$\therefore\ x''y''-x'y' = 0;$$

$\therefore$ subtracting and adding $x''y'$, we have

$$x''y''-x''y'-x'y'+x''y' = 0,$$

$$\text{or } x''(y''-y')+(x''-x')y' = 0;$$

$$\therefore\ \frac{y''-y'}{x''-x'} = -\frac{y'}{x''};$$

therefore substituting in (1), we have

$$y-y' = -\frac{y'}{x''}(x-x').$$

Suppose now, that the points (x', y), and (x'', y'') coincide; then the secant becomes a tangent, and $x''=x'$, $y''=y'$; therefore the equation to the tangent is

$$y-y' = -\frac{y'}{x'}(x-x') \ldots\ldots\ldots (2).$$

This equation, when reduced, assumes the very simple form

$$xy'+yx' = 2m^2 \ldots\ldots\ldots\ldots (3).$$

181. Cor. If a tangent applied at any point be produced to meet the asymptotes, the area of the triangle thus cut off is constant, (fig. 64.)

For in the equation $xy'+yx' = 2m^2$

$$\text{Let } y=0;\ \therefore x = \frac{2m^2}{y'} = CT,$$

$$x=0;\ \therefore y = \frac{2m^2}{x'} = Ct;$$

$$\therefore \tfrac{1}{2}\,CT\,.\,Ct\,.\sin x, y = \tfrac{1}{2}\sin x, y\,.\frac{4m^4}{x'y'};$$

$$\therefore \text{area of triangle } TCt = \sin x, y\,.\frac{2m^4}{m^2}$$

$$= 2m^2.\sin x, y,$$

which is a constant quantity; whence the truth of the Cor.

182. Prop. 5. The asymptotes, and one point in the curve being given, to find the direction and magnitude of the principal diameters.

(1) To find their direction.

Bisect the angle, and the supplement of the angle, contained by the asymptotes; then the direction of the principal diameters is known.

(2) To find their magnitude.

The co-ordinates of the given point being x', y', we have

$$a^2+b^2 = 4x'y' \ldots\ldots\ldots\ldots (1),$$

$$\text{and } \pm\frac{a}{b} = \tan\tfrac{1}{2}x, y \ \ldots\ldots (2),$$

$$\text{whence, } a = \pm 2\cos\tfrac{1}{2}x, y\sqrt{x'y'},$$

$$b = \pm 2\sin\tfrac{1}{2}x, y\sqrt{x'y'};$$

$\therefore$ the position and magnitude of the principal diameters are determined.

183. The principal properties of the ellipse, hyperbola, and

parabola, having now been deduced from their respective equations, we propose in the next place, to shew how such properties as are common to the three curves, may be investigated by means of their general equation. Previously, however, to entering upon this inquiry, it may be useful to bring under review the various equations, by which lines of the second order are characterized. These equations may be arranged as follows:

I. Algebraical equations.

(1) Let the curve be referred to its centre and principal diameter, then

$$\text{in the ellipse, } y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\text{hyperbola, } y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

$$\text{circle, } y^2 = a^2 - x^2$$

$$\text{equilateral hyperbola, } y^2 = x^2 - a^2$$

(2) Let the curve be referred to the principal diameter, and the tangent at its vertex, then

$$\text{in the ellipse, } y^2 = \frac{b^2}{a^2}(2ax - x^2)$$

$$\text{hyperbola, } y^2 = \frac{b^2}{a^2}(2ax + x^2)$$

$$\text{parabola, } y^2 = 2px$$

$$\text{circle, } y^2 = 2ax - x^2$$

$$\text{equilateral hyperbola, } y^2 = x^2 - 2ax$$

The general equation is $y^2 = mx + nx^2$

which represents an ellipse, an hyperbola, a parabola, or a circle, according as n is *negative*, *positive*, 0, or $= -1$.

When the curve is referred to *any* system of conjugate diameters, the equations will be of the same form, and may be obtained from those above given, by merely changing a and b into a' and b'.

II. Polar equations.

(1) Let the centre be assumed as the pole, then

$$\text{in the ellipse, } r = a\sqrt{\frac{1-e^2}{1-e^2\cos^2\theta}}$$

$$\text{hyperbola, } r = a\sqrt{\frac{e^2-1}{e^2\cos^2\theta-1}}$$

(2) Let either focus, S for example, be the pole, then

$$\text{in the ellipse, } r = a\,.\,\frac{1-e^2}{1+e\cos\omega}$$

$$\text{hyperbola, } r = a\,.\,\frac{e^2-1}{1+e\cos\omega}$$

$$\text{parabola, } r = \frac{p}{1+\cos\omega}$$

III. Equations between the radius vector and the perpendicular dropped upon the tangent.

(1) Let the centre be the pole; then, if P be the perpendicular, and ρ the radius vector,

$$\text{in the ellipse, } P = \frac{ab}{\sqrt{(a^2+b^2-\rho^2)}}$$

$$\text{hyperbola, } P = \frac{ab}{\sqrt{\{\rho^2-(a^2-b^2)\}}}$$

(2) Let the focus S be the pole, then, if P' be the perpendicular, and r the radius vector,

$$\text{in the ellipse, } P' = b\sqrt{\frac{r}{2a-r}}$$

$$\text{hyperbola, } P' = b\sqrt{\frac{r}{2a+r}}$$

$$\text{parabola, } P' = \sqrt{\tfrac{1}{2}pr}$$

The equation to a secant drawn through any two points (x', y') and (x'', y'') is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \text{ ...... (1).}$$

Now these points being in the hyperbola,

$$x'y' = m^2 = x''y'';$$

$$\therefore x''y'' - x'y' = 0;$$

$\therefore$ subtracting and adding $x''y'$, we have

$$x''y'' - x''y' - x'y' + x''y' = 0,$$

$$\text{or } x''(y'' - y') + (x'' - x')y' = 0;$$

$$\therefore \frac{y'' - y'}{x'' - x'} = -\frac{y'}{x''};$$

therefore substituting in (1), we have

$$y - y' = -\frac{y'}{x''}(x - x').$$

Suppose now, that the points (x', y), and (x'', y'') coincide; then the secant becomes a tangent, and $x'' = x'$, $y'' = y'$; therefore the equation to the tangent is

$$y - y' = -\frac{y'}{x'}(x - x') \text{ .... (2).}$$

This equation, when reduced, assumes the very simple form

$$xy' + yx' = 2m^2 \text{ .... (3).}$$

181. Cor. If a tangent applied at any point be produced to meet the asymptotes, the area of the triangle thus cut off is constant, (fig. 64.)

For in the equation $xy' + yx' = 2m^2$

$$\text{Let } y=0;\ \therefore x = \frac{2m^2}{y'} = CT,$$

$$x=0;\ \therefore y = \frac{2m^2}{x'} = Ct;$$

$$\therefore \tfrac{1}{2}\, CT \,.\, Ct \,.\, \sin x, y = \tfrac{1}{2} \sin x, y \,.\, \frac{4m^4}{x'y'};$$

$$\therefore \text{area of triangle } TCt = \sin x, y \,.\, \frac{2m^4}{m^2}$$

$$= 2m^2 . \sin x, y,$$

which is a constant quantity; whence the truth of the Cor.

182. PROP. 5. The asymptotes, and one point in the curve being given, to find the direction and magnitude of the principal diameters.

(1) To find their direction.

Bisect the angle, and the supplement of the angle, contained by the asymptotes; then the direction of the principal diameters is known.

(2) To find their magnitude.

The co-ordinates of the given point being x', y', we have

$$a^2 + b^2 = 4x'y' \ldots\ldots\ldots\ldots (1),$$

$$\text{and } \pm \frac{a}{b} = \tan \tfrac{1}{2} x, y \ \ldots\ldots (2),$$

$$\text{whence, } a = \pm 2 \cos \tfrac{1}{2} x, y \sqrt{x'y'},$$

$$b = \pm 2 \sin \tfrac{1}{2} x, y \sqrt{x'y'};$$

$\therefore$ the position and magnitude of the principal diameters are determined.

183. The principal properties of the ellipse, hyperbola, and

$$a(y'-y'')(y'+y'')+b\{x'(y'-y'')+y''(x'-x'')\}$$
$$+c(x'+x'')(x'-x'')+d(y'-y'')+e(x'-x'')=0,$$

or collecting the terms that involve $y'-y''$, $x'-x''$,

$$(y'-y'')\{a(y'+y'')+bx'+d\}+(x'-x'')\{by''+c(x'+x'')+e\}=0,$$

$$\text{whence } \frac{y'-y''}{x'-x''}=-\frac{by''+c(x'+x'')+e}{a(y'+y'')+bx'+d},$$

this value of $\frac{y'-y''}{x'-x''}$ being substituted in (1), the equation to the secant is determined.

Suppose, now, that the points of section coincide,

then $x'=x''$, and $y'=y''$,

and the secant becomes a tangent.

Hence, the equation sought is

$$y-y'=-\frac{by'+2cx'+e}{2ay'+bx'+d}(x-x').$$

188. Cor. 1. Since $ay'^2+bx'y'+cx'^2+dy'+ex'+f=0$, the equation to the tangent becomes, on reduction,

$$\{2ay'+bx'+d\}y+\{2cx'+by'+e\}x+dy'+ex'+2f=0.$$

Note. The equation to the tangent is deducible from the equation to the curve, by the following rule:

"Write down the general equation to the curve in two lines, one below the other: in the first line change x^2, y^2, xy, into xx', yy', xy', and in the second change, x^2, y^2, xy, x, y into xx', yy', yx', x', y', add the two lines together, and their sum will be the equation sought."

189. Cor. 2. If $b=0$, or the curve be referred to axes which are parallel to a system of conjugate diameters, then the equation in the last Cor. becomes

$$(2ay'+d)y+(2cx'+e)x+dy'+ex'+2f=0.$$

If, moreover, the origin be on the curve, then $f=0$, and the equation becomes

$$(2ay'+d)\,y+(2cx'+e)\,x+dy'+ex'=0.$$

190. **Prop. 4.** To draw a tangent to a line of the second order from a given point without it.

Let Q (α, β) be the given point.

When the axes originate on the curve, and are parallel to a system of conjugate diameters, the equation to the tangent drawn through a given point (x', y') in the curve, is

$$(2ay'+d)\,y+(2cx'+e)\,x+dy'+ex'=0 \ldots . (1).$$

Now, since the given point Q is upon the tangent, the equation will be satisfied, if α, β be substituted for x, y;

$$\therefore\ (2ay'+d)\,\beta+(2cx'+e)\,\alpha+dy'+ex'=0 \ldots . (2),$$

or, altering the arrangement of the terms, since x', y' are now supposed to be unknown,

$$(2a\beta+d)\,y'+(2c\alpha+e)\,x'+d\beta+e\alpha=0 \ldots . (3).$$

But, (x', y') being a point in the curve, we have

$$ay'^2+cx'^2+dy'+ex'=0 \ldots \ldots \ldots (4).$$

Hence, x' and y' may be found by elimination between (3) and (4).

It is more convenient, however, to determine the points of contact, as in Art. 118, by constructing the loci of the foregoing equations.

Now, the latter of these is the curve itself, and as the former represents a straight line, it must be the line joining the points of contact. This line being constructed, we arrive at the solution of the proposed question.

Cor. The line Pp, which joins the points of contact, will vary with the position of the point Q; let us suppose, however, that Pp always passes through a given point (α', β');

Then, the equation to Pp will become, on substitution of α', β' for x', y' in (2)

$$(2a\beta' + d)\beta + (2c\alpha' + e)\alpha + d\beta' + e\alpha' = 0,$$

which is the equation to a straight line.

Whence is derived the following theorem: "if through a given point, within a line of the second order, any number of chords be drawn, and tangents be applied at their extremities, the points of intersection of each pair of tangents will lie in a straight line."

In like manner, the converse of the theorem may be proved, that "if from the different points of a straight line, given in position, without a line of the second order, tangents be drawn to the curve, the lines joining the points of contact of each pair of tangents, will all pass through the same point."

191. Prop. 5. In a line of the second order, to find a point M in the principal diameter AX, such that drawing any chord whatever PMp, and joining AP, Ap, the angle PAp so formed may be a right angle.

The curve being referred to AX, and to the tangent at its extremity, as rectangular axes, we have

$$y^2 = mx + nx^2 \ldots\ldots (1).$$

From the vertex A draw two chords AP, Ap, at right angles to one another, and join Pp cutting AX in M: the aim of the problem is to find the value of AM.

The equation to the chord AP being $y = ax$,

that to Ap will be $y = -\frac{1}{a}x$;

therefore, eliminating y between each of these equations and equation (1), we have

$$mx+nx^2=a^2x^2, \text{ or } a^2=\frac{m+nx}{x},$$

$$mx+nx^2=\frac{x^2}{a^2}, \text{ or } a^2=\frac{x}{m+nx};$$

therefore, equating these values of a^2,

$$(m+nx)^2=x^2,$$

$$\text{or } m=(1-n)\,x;$$

$$\therefore\ x=\frac{m}{1-n};$$

whence the point M is known.

192. **Cor.** In the ellipse, $m=\frac{2b^2}{a}$, and $n=-\frac{b^2}{a^2}$;

$$\therefore\ x=\frac{2b^2}{a}\times\frac{a^2}{a^2+b^2}=\frac{2ab^2}{a^2+b^2}.$$

In the hyperbola, $x=\frac{2ab^2}{a^2-b^2}$.

In the parabola, $x=m=$ the latus rectum.

193. **Prop. 6.** If through a given point within or without a line of the second order, two straight lines of known inclination to the axes be drawn, the rectangles contained by the segments of each will have a given ratio to one another.

Let $P\,(\alpha,\beta)$ be the given point, Mm and Nn any two lines drawn through P to meet the curve; it is required to prove that $\frac{MP\,.\,Pm}{NP\,.\,Pn}$ is a given quantity.

The equation to the curve, when referred to any diameter, and the tangent at its vertex, is

$$y^2=mx+nx^2\ .\ .\ .\ .\ .\ .\ .\ .\ (1).$$

If the curve be now transformed to polar co-ordinates, originating at P, we have, by the formulas of (78), if $PM = r$,

$$x = a + r\frac{\sin r, y}{\sin x, y},$$

$$y = \beta + r\frac{\sin r, x}{\sin x, y}.$$

Substituting these values of x and y in (1), and arranging the result,

$$r^2 - \frac{(m+2an)\sin r,y - 2b\sin r,x}{\sin^2 r,x - n\sin^2 r,y}.\sin x,y.r + \frac{b^2 - a(m+na).\sin^2 x,y}{\sin^2 r,x - n\sin^2 r,y} = 0;$$

therefore, by the theory of equations,

$$PM.Pm = \frac{\sin^2 x,y}{\sin^2 r,x - n\sin^2 r,y}\{b^2 - a(m+na)\}.$$

In like manner, if $PN = r'$,

$$PN.Pn = \frac{\sin^2 x,y}{\sin^2 r',x - n\sin^2 r',y}\{b^2 - a(m+na)\}.$$

Hence $\dfrac{PM.Pm}{PN.Pn} = \dfrac{\sin^2 r',x - n\sin^2 r',y}{\sin^2 r,x - n\sin^2 r,y}$,

which is a given quantity, since the angles of inclination r, x, and r, y; r', x, and r', y are supposed to be known.

194. Cor. 1. Let the given inclination be equal to that of the axes, that is, let Mm and Nn be parallel to AY, AX, respectively.

Then, since $\angle r', x = 0$, and $\angle r', y = x, y$,

$\angle r, x = x,y$, and $\angle r, y = 0$,

we have $\dfrac{PM.Pm}{PN.Pn} = \dfrac{-n\sin^2 x,y}{\sin^2 x,y} = -n.$

But in the ellipse or hyperbola, $-n = \pm \frac{b'^2}{a'^2}$;

$$\therefore \frac{PM . Pm}{PN . Pn} = \pm \frac{b'^2}{a'^2}.$$

Cor. 2. Let the point be within the curve, and suppose Nn to be a diameter bisecting the chord Mm; we have, then, by the last Cor.

$$PM^2 = \pm \frac{b'^2}{a'^2} . PN . Pn,$$

which is no other than the equation to the ellipse and hyperbola, when referred to conjugate diameters (100).

The general equation to the parabola (164) may, in like manner, be shewn to be a particular case of the proposition under consideration.

Cor. 3. Let the secants move parallel to themselves, until they become tangents Pp, Pq; then

$$\frac{Pp^2}{Pq^2} = \frac{b'^2}{a'^2}; \text{ or } \frac{Pp}{Pq} = \frac{b'}{a'}.$$

In the circle, $b' = a'$; $\therefore Pp = Pq$.

195. Prop. 7. To find the locus of the point of intersection of two tangents to a line of the second order, when the product of the trigonometrical tangents of their inclination to the axis is supposed constant.

$$\text{Let } y = ax + b \ldots\ldots (1)$$

be a straight line which cuts a line of the second order,

$$y^2 = mx + nx^2 \ldots (2).$$

referred to the principal diameter, and the tangent at its vertex.

The abscissas of the points of intersection will be found by eliminating y between these equations; we have, therefore,

$$(ax+b)^2=mx+nx^2,$$

or, transposing and arranging,

$$(a^2-n)\,x^2+(2ab-m)\,x+b^2=0 \;\ldots\ldots\; (3).$$

Let the points of section be now supposed to coincide; the secant will then become a tangent, and the roots of equation (3) being equal, the equation will be a complete square.

$$\text{Hence, } 4(a^2-n)\,b^2=(2ab-m)^2,$$

$$\text{or } m^2-4b\,(am-bn)=0;$$

therefore, substituting for b its value $y-ax$, from equation (1), we have

$$m^2-4\,(y-ax)\,\{am-n\,(y-ax)\}=0;$$

therefore, reducing and arranging the result by the powers of a,

$$a^2-\frac{my+2nx}{mx+nx^2}\cdot a+\frac{ny^2+\frac{1}{4}m^2}{mx+nx^2}=0.$$

In like manner, if the straight line

$$y=a'x+b' \text{ touch the curve,}$$

$$a'^2-\frac{my+2nx}{mx+nx^2}\,a'+\frac{ny^2+\frac{1}{4}m^2}{mx+nx^2}=0.$$

Hence, by the theory of equations,

$$aa'=\frac{ny^2+\frac{1}{4}m^2}{mx+nx^2};$$

$$\therefore\; ny^2-naa'x^2-maa'x+\tfrac{1}{4}m^2=0 \;\ldots\ldots\; (4).$$

Therefore, the locus required is an hyperbola or ellipse, according as the constant quantity aa' is positive or negative.

When $n=0$, or when the curve is a parabola, the equation becomes

$$-maa'x+\tfrac{1}{4}m^2=0;$$

$$\therefore\ x=\frac{m}{4aa'};$$

therefore, the locus in this case is a straight line, at right angles to the axis.

196. Cor. 1. The curve being supposed an ellipse or hyperbola, let $aa'=\mp\frac{b'^2}{a'^2}$, or the tangents be parallel to a system of conjugate diameters.

Then equation (4) becomes

$$ny^2\pm n\frac{b'^2}{a'^2}x^2\pm m\frac{b'^2}{a'^2}x+\tfrac{1}{4}m^2=0,$$

$$\text{or } y^2\pm\frac{b'^2}{a'^2}x^2\pm\frac{m}{n}\frac{b'^2}{a'^2}x+\tfrac{1}{4}\frac{m^2}{n}=0,$$

$$\text{but } n=\mp\frac{b'^2}{a'^2},\ \text{and } m=\frac{2b'^2}{a'};$$

therefore, by substitution,

$$y^2\pm\frac{b'^2}{a'^2}x^2\mp\frac{2b'^2}{a'}x\mp b'^2=0,$$

Hence, in the ellipse and hyperbola, the locus of the intersection of tangents drawn parallel to a system of conjugate diameters, is an ellipse or hyperbola.

197. Cor. 2. If $aa'=-1$, or the tangents be at right angles to one another,

then, equation (4) becomes

$$y^2+x^2+\frac{m}{n}x+\tfrac{1}{4}m^2=0,$$

which is the equation to a circle.

Hence, in the ellipse and hyperbola, the locus of the intersection of pairs of tangents, which form a right angle, is a circle.

In the parabola, $x=-\frac{m}{4}$,

whence the locus, in this case, is the directrix.

◆

CHAP. IX.

ON THE GENERAL DISCUSSION OF LINES OF THE SECOND ORDER.

In this chapter, we shall suppose the axes to be inclined at any angle whatever, unless the contrary be specified.

198. **Prop. 1.** To find when a line of the second order intersects the axes.

The general equation being

$$ay^2+bxy+cx^2+dy+ex+f=0,$$

Let $y=0$, that is, let the curve cut the axis of x,

Then, $cx^2+ex+f=0$;

$$\therefore\ x=-\frac{e}{2c}\pm\frac{1}{2c}\sqrt{(e^2-4cf)},$$

which are the abscissas of the points of intersection.

In order, therefore, that the intersection may take place, e^2 must $>4cf$.

If $e^2<4cf$, the values of x are imaginary, and the curve does not meet the axis of x.

If $e^2=4cf$, the two values of x are equal, hence the points of section coincide, and the curve *touches* the axis of x.

In like manner it may be shewn, that according as $d^2 >$ or $<$ or $= 4af$, the curve intersects, falls without, or touches the axis of y.

199. Prop. 2. To find when the curve intersects either of the diameters BR, DS, fig. 62.

The general equation being resolved in terms of x and y successively, we have

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\sqrt{\{Ax^2+2Bx+C\}},$$

$$x = -\frac{by+e}{2c} \pm \frac{1}{2c}\sqrt{\{Ay^2+2B'y+C'\}}.$$

We shall first consider the former of these equations.

When the curve intersects the diameter $-\frac{bx+d}{2a}$, the co-ordinates of the curve and of the diameter, will be identical. Hence, the irrational part of the equation must vanish, and therefore,

$$Ax^2+2Bx+C=0;$$

$$\therefore\ x = -\frac{B}{A} \pm \frac{1}{A}\sqrt{(B^2-AC)}.$$

Assume $AO = -\frac{B}{A}$, and OD, OD', on opposite sides of O, each $= \frac{1}{A}\sqrt{(B^2-AC)}$, then, the two values of x are AD', AD.

In order, therefore, that the curve may intersect the diameter BR, B^2 must $> AC$.

If $B^2 < AC$, then the curve does not meet the diameter.

If $B^2 = AC$, then the curve touches the diameter.

It may be proved, in the same manner, that according as $B'^2 >$ or $<$ or $= AC'$, the curve intersects, falls without, or touches the diameter DS.

200. Cor. If $AD = \alpha$, and $AD' = \beta$, then the equation

$$Ax^2 + 2Bx + C = 0 \text{ becomes}$$

$$A(x-\alpha)(x-\beta) = 0.$$

201. We now proceed to investigate the figure of the various curves comprised in the equation

$$ay^2 + bxy + cx^2 + dy + ex + f = 0.$$

We have, as before,

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a} \sqrt{\{Ax^2 + 2Bx + C\}},$$

$$x = -\frac{by+e}{2c} \pm \frac{1}{2c} \sqrt{\{Ay^2 + 2B'y + C'\}}.$$

The nature of the curve depends, it is manifest, on the irrational part of these equations. Now, x or y may be so assumed as to render the first term, in each case, greater than the sum of the two remaining terms. The sign, therefore, of the trinomials $Ax^2 + 2Bx + C$, and $Ay^2 + 2B'y + C'$, will depend on that of the first term, or, since x^2 and y^2 are necessarily positive, on that of A.

We are thus led to divide lines of the second order into three species, characterized by the quantity A being *negative*, *positive*, or *equal to nothing*.

In Art. 90, we arrived at the same conclusion, from different principles.

202. Prop. 3. To trace the curve when A is *negative*, fig. 63.

(1) Let the roots of the equation $A(x-\alpha), (x-\beta) = 0$ be *real*.

If $x < \alpha$, or $< AD$, it must also be $< \beta$, or $< AD'$; each factor then, $x-\alpha$, $x-\beta$, being negative, their product is positive, and, therefore, $A(x-\alpha)(x-\beta)$ is negative.

Hence, the values of y are *imaginary*, and no part of the curve lies between A and D.

If $x>\alpha$, but $< \beta$, that is, if (x) be of intermediate value between AD and AD', then, $x-\alpha$ being positive, and $x-\beta$ negative, their product is negative, and therefore,

$$A(x-\alpha)(x-\beta)$$

is positive.

Hence, for each assumed value of x, there are two real values of y, equal and with opposite signs.

If $x > \alpha'$, or if x be *negative*, the values of y are in each case imaginary, and, therefore, the curve cannot extend to the *right* of D', or to the *left* of A. The curve, therefore, will be *limited*, in the direction AX, by the parallels Dd, $D'd'$.

When the roots α, β are *equal*, then

$$y = -\frac{bx+d}{2a} \pm \frac{x-\alpha}{2a}\sqrt{A},$$

which is an *imaginary* quantity.

If $x=\alpha$, then $y = -\dfrac{b\alpha+d}{2a}$,

hence, *in this case the curve is reduced to a point.*

(2) Let the roots be *imaginary*, and suppose them to be

$$\alpha' + \beta'\sqrt{-1}, \text{ and } \alpha' - \beta'\sqrt{-1}.$$

Then,

$$\{x-(\alpha'+\beta'\sqrt{-1})\}\,.\,\{x-(\alpha'-\beta'\sqrt{-1})\} = (x-\alpha')^2 + \beta'^2,$$

which is positive; therefore, $A\{(x-a')^2+\beta'^2\}$ being negative, the values of y are imaginary. Hence, in this case, the *curve itself* is an imaginary line.

We have hitherto considered only the values of y in terms of x, but the discussion of x in terms of y will be in all respects similar.

Let AE, AE', be the roots of the equation

$$y^2+\frac{2B'}{A}y+\frac{C'}{A}=0;$$

then, if Ee, $E'e'$, be drawn parallel to AX, it may be shewn, as in the former case, that the curve is *limited*, in the direction AY', by the lines Ee, $E'e'$.

The curve thus traced is the ellipse, and is wholly comprised between the parallels Dd, $D'd'$, and Ee, $E'e'$.

203. Cor. 1. Since b^2 is necessarily positive, it follows, that in the equation to the ellipse, (a) and (c) *can neither of them* $=0$, *and must always have the same signs.*

204. Cor. 2. The general equation to the circle being

$$y^2+x^2+2xy\cos x,y-2(y'+x'\cos x,y)\,y-2(x'+y'\cos x,y)\,x$$
$$+y'^2+x'^2+2x'y'\cos x,y-r^2=0,$$

we have, on comparing it with the general equation of the second degree,

$$a=c=1,\ \text{ and }\ b=2\cos x,y;$$

$$\therefore\ b^2-4ac=4\cos x,y-4=-4\sin^2 x,y,$$

which is a negative quantity.

The circle, therefore, is a species of the ellipse.

When the axes are rectangular, $\cos x,y=0$; hence the general equation of the second degree represents a circle, *when* $a=c$, *and the term involving* xy *vanishes.*

The equation becomes, in this case,

$$a(y^2+x^2)+dy+ex+f=0,$$

$$\text{or } y^2+x^2+\frac{d}{a}y+\frac{e}{a}x+\frac{f}{a}=0,$$

which (62) is the equation to a circle, the co-ordinates of whose centre are $-\frac{d}{2a}$, $-\frac{e}{2a}$, and whose radius

$$=\frac{1}{2a}\sqrt{\{d^2+e^2-4af\}}.$$

Observation. It appears from the preceding discussion, that the varieties of an ellipse are *a circle*, *a point*, and an *imaginary line*.

205. PROP. 4. To trace the curve when A is *positive*, fig. 64.

The roots α, β, may be supposed in this as in the former case, to be either *real* or *imaginary*.

(1) Let them be real.

Let $x < \alpha$, or $< AD$, it must also $< \beta$, or $< AD'$, therefore $x-\alpha$, $x-\beta$ being each positive, the quantity $A(x-\alpha)(x-\beta)$ is also positive. Hence the values of y are *real*. When x is *negative*, the values of y still remain real, for the product of the factors is, in that case, $(x+\alpha)(x+\beta)$; $\therefore A(x+\alpha)(x+\beta)$ is positive.

How great soever the negative value of x be taken, there are always two *real* corresponding values of y. Hence, the curve extends indefinitely to the *left* of D.

If $x > \alpha$, but $< \beta$, that is, if x be of intermediate value between AD and AD', then $x-\alpha$ being positive, and $x-\beta$ negative, the quantity $A(x-\alpha)(x-\beta)$ is *negative*, and, therefore, the values of y are imaginary.

No part of the curve, therefore, lies between D and D'.

If $x > \beta$, and, therefore, $> \alpha$, then $x - \alpha$, $x - \beta$ being each positive, $A(x-\alpha)(x-\beta)$ is so likewise, therefore the values of y are real.

How great soever x be taken, the quantity $A(x-\alpha)(x-\beta)$ still continues positive; therefore the values of y are real, and the curve extends indefinitely to the *right* of D'.

If $\alpha = \beta$

$$\text{Then } y = -\frac{bx+d}{2a} \pm \frac{x-\alpha}{2a}\sqrt{A}.$$

therefore the values of y are real, and represent two straight lines which *intersect one another*, since the coefficients of x are different, being $-\dfrac{b+\sqrt{A}}{2a}$, and $-\dfrac{b-\sqrt{A}}{2a}$.

(2) Let the roots be imaginary, in which case the diameter does not meet the curve.

Then, as in Prop. 3, their product $=(x-\alpha')^2+\beta'^2$; therefore, $A\{(x-\alpha')^2+\beta'^2\}$ is positive, and the values of y are *real*. The curve in this case consists of two infinite branches, the one *above*, the other *below*, the diameter BR. Fig. 67 *a*.

The curve traced above is the hyperbola.

206. Cor. 1. Since b^2 is necessarily positive, either (a) or (c) or both of them may $=0$: if $b=0$, then (a) and (c) must have different signs. Hence, the equation to the hyperbola may assume any one of the three following forms:

$$(1)\quad bxy + dy + ex + f = 0.$$

$$(2)\quad ay^2 - cx^2 + dy + ex + f = 0.$$

$$(3)\quad cx^2 - ay^2 + dy + ex + f = 0.$$

207. Cor. 2. Let the axes be supposed rectangular, and in either of the two last equations, let $a = -c$.

Then the equation becomes

$$a(y^2 - x^2) + dy + ex + f = 0;$$

therefore dividing by (a), and transposing,

$$\left(y^2 + \frac{d}{a}y\right) - \left(x^2 - \frac{e}{a}x\right) = -\frac{f}{a},$$

adding $\frac{d^2 - e^2}{4a^2}$ to each, we have

$$\left(y + \frac{d}{2a}\right)^2 - \left(x - \frac{e}{2a}\right)^2 = \frac{d^2 - e^2 - 4af}{4a^2},$$

or if $y + \frac{d}{a} = y'$, $x - \frac{e}{a} = x'$, and $\frac{d^2 - e^2 - 4af}{4a^2} = r^2$,

$$\text{then } y'^2 - x'^2 = r^2,$$

which is the equation to the *equilateral* hyperbola (95).

Observation. The varieties of the hyperbola are *two straight lines that cut one another*, and the *equilateral* hyperbola.

208. Prop. 5. To trace the curve when $A = 0$, fig. 68.

In this case $y = -\frac{bx + d}{2a} \pm \frac{1}{2a}\sqrt{(2Bx + C)}$.

When the diameter meets the curve,

$$2Bx + C = 0; \quad \therefore\ x = -\frac{C}{2B}.$$

Assume $AD = -\frac{C}{2B}$, and draw Dd parallel to AY; then if $AD = a$,

$$y = -\frac{bx + d}{2a} \pm \frac{1}{2a}\sqrt{2B(x - a)}.$$

Since the coefficient B may be positive or negative, or $= 0$.

(1) Let it be positive.

When $x < a$, $2B(x-a)$ is negative; therefore the values of y are imaginary, and no part of the curve lies between A and D.

When $x > a$, $2B(x-a)$ is positive, and for each assumed value of x, there are two equal values of y, with opposite signs.

How great soever x be taken, the values of y remain real; hence the curve consists of two infinite arcs dZ, dz extending to the *right* of D.

(2) Let B be negative.

$$\text{Then if } x < a,\ 2B(x-a) \text{ is positive,}$$
$$x > a,\ 2B(x-a) \text{ is negative,}$$

The curve, therefore, in this case extends indefinitely to the left of A, and is limited towards the right by the line Dd.

(3) Let $B=0$.

$$\text{Then } y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\sqrt{C} = -\frac{b}{2a}x - \frac{d}{2a} \pm \frac{1}{2a}\sqrt{C},$$

which is the equation to *two parallel straight lines.*

$$\text{If, now, } C=0, \text{ then } y = -\frac{bx+d}{2a},$$

which is the equation to the diameter BR.

If C be negative, then the values of y are imaginary.

The curve thus traced is the parabola.

209. Cor. 1. Since

$$b^2-4ac=0, \text{ or } b^2=4ac;\ \therefore b=\pm 2\sqrt{(ac)};$$

therefore the general equation becomes

$$(a^{\frac{1}{2}}y \pm c^{\frac{1}{2}}x)^2 + dy + ex + f = 0.$$

Hence, *when the first three terms form a complete square, the equation represents a parabola.*

210. Cor. 2. Since $b^2-4ac=0$, if $b=0$, then (a) or (c) must $=0$, and conversely, if a or $c=0$, then must $b=0$.

The equation to the parabola may, therefore, assume either of the two following forms:

$$(1)\quad ay^2 + dy + ex + f = 0.$$

$$(2)\quad cx^2 + dy + ex + f = 0.$$

Observation. The varieties of the parabola are *two parallel straight lines, one straight line,* and *an imaginary line.*

211. Prop. 6. To determine the form of the general equation, when it represents any one of the varieties of the three curves.

I. Let A be negative.

(1) Suppose the roots α, β to be equal;

$$\text{Then } y = -\frac{bx+d}{2a} \pm \frac{x-\alpha}{2a}\sqrt{A};$$

$$\therefore\ \{2ay + bx + d\}^2 - A(x-\alpha)^2 = 0.$$

Which, since A is negative, must be the sum of two positive quantities.

Now this sum cannot, it is plain, $=0$, unless each of its parts $= 0$;

$$\therefore\ 2ay + bx + d = \quad, \text{ and } x - \alpha = 0;$$

$$\therefore\ x = \alpha, \text{ and } y = -\frac{b\alpha + d}{2a},$$

which is the equation to a point.

The general equation, therefore, represents a point, when it is the sum of two positive quantities, each of which is a function of x, y.

(2) Let the roots be imaginary;

$$\therefore\ y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\surd A\{(x-\alpha')^2 + \beta'^2\};$$

$$\therefore (2ay + bx + d)^2 - A(x - \alpha')^2 - A\beta'^2 = 0,$$

which, as A is negative, is the sum of three squares, the *last being independent of* x *and* y; whatever value, therefore, be assigned to x and y, that sum cannot $= 0$. *Hence, the equation represents an imaginary line, when it is the sum of three squares, the last of which is a constant quantity.*

II. Let A be positive.

(1) Let the roots be equal,

Then, as before, $\{2ay + bx + d\}^2 - A(x - \alpha)^2 = 0$.

And, as A is positive, this quantity represents the difference of two squares, and is, therefore, equal to the product of the two factors

$$\{2ay + bx + d\} + \sqrt{A}.(x - \alpha),$$

$$\{2ay + bx + d\} - \sqrt{A}.(x - \alpha).$$

Each of which must $= 0$;

$$\therefore y = -\frac{b + \sqrt{A}}{2a}x - \frac{d - \alpha\sqrt{A}}{2a},$$

$$y = -\frac{b - \sqrt{A}}{2a}x - \frac{d + \alpha\sqrt{A}}{2a},$$

which are the equations to two straight lines that intersect each other. Hence, the *general equation represents two straight lines, that intersect, when it is the difference of two squares, each of which is a function of x, y.*

(2) When the roots are imaginary, then the curve exists, as was shewn above.

III. Let $A = 0$.

If B at the same time $= 0$, then

$$y = -\frac{bx + d}{2a} \pm \frac{1}{2a}\sqrt{C},$$

$$\text{or } \{2ay + bx + d\}^2 - C = 0.$$

Here C may be positive or negative, or $=0$.

(1) Let C be positive.

Then the equation is the difference of two squares, and equal, therefore, to the product of the two factors

$$\{2ay+bx+d\}+C$$
$$\{2ay+bx+d\}-C$$

Each of which being supposed $=0$,

$$y=-\frac{b}{2a}x-\frac{d+C}{2a},$$

$$y=-\frac{b}{2a}x-\frac{d-C}{2a}.$$

These are the equations to two straight lines, which, as the coefficient of x is the same in each, are parallel.

(2) Let $C=0$, then the equation becomes

$$\{2ay+bx+d\}^2=0,$$

$$\text{or } y=-\frac{bx+d}{2a},$$

which represents *one* straight line.

(3) Let C be negative, then the equation becomes the sum of two squares, one of which being independent of x, y, cannot $=0$; therefore the curve is imaginary.

212. In order to present a connected view of the various circumstances relating to the discussion of the general equation, we shall briefly recapitulate the foregoing results.

I. DISCUSSION of the curve.

1. The general equation being

$$ay^2+bxy+cx^2+dy+ex+f=0.$$

The curve will be an ellipse, an hyperbola, or a parabola, according as

$$b^2-4ac < \text{ or } > \text{ or } = 0.$$

2. The curve intersects, touches, or falls without, the axis of x or y, according as

$$e^2-4cf \text{ or } d^2-4af > \text{ or } =< 0.$$

3. The curve intersects, touches, or falls without,

(1) The diameter $y=-\dfrac{bx+d}{2a}$; according as

$$(bd-2ae)^2-(b^2-4ac)(d^2-4af) >= \text{ or } < 0.$$

(2) The diameter $x=-\dfrac{by+e}{2c}$; according as

$$(be-2cd)^2-(b^2-4ac)(e^2-4cf) > = \text{ or } <0.$$

II. VARIETIES of the three curves.

1. Let $b^2-4ac<0$, or the curve be an ellipse.

Then the general equation represents

(1) *A point,* when it is the sum of two positive quantities, each of which is a function of x, y.

(2) *An imaginary line,* when it is the sum of three squares, the last of which is a constant quantity.

(3) *A circle,* when $a=c$, and $b=2a\cos x, y$;

or, if the axes be rectangular, when $a=c$, and $b=0$.

2. Let $b^2-4ac>0$, or the curve be an hyperbola.

Then the general equation represents

(1) *Two straight lines that intersect one another,* when it is the difference of two squares, each of which is a function of x, y.

(2) *An equilateral hyperbola,* when, the axes being rectangular, $a=-c$, and $b=0$.

3. Let $b^2-4ac=0$, or the curve be a parabola.

Then the general equation represents

(1) *Two parallel lines,* when it is the difference of two squares, *one of which is a constant quantity.*

(2) *One straight line,* when it is a perfect square.

(3) *An imaginary line,* when it is the sum of two squares, *one of which is constant.*

213. In applying the principles above laid down to the discussion of numerical examples, it will be found convenient, in each case, to conduct the investigation as follows:

(1) Resolve the equation to the curve in terms of x and of y, so that the *form* of the resulting equations may be

$$y = mx + n \pm \sqrt{(Ax^2 + 2Bx + C)},$$

$$x = m'y + n' \pm \sqrt{(Ay^2 + 2B'y + C')}.$$

(2) Find where the curve cuts the axes of x and y.

(3) Draw the diameters $y' = mx + n$,

$$x' = m'y + n',$$

and thence determine the co-ordinates of the centre, if there be one.

(4) Find where these diameters intersect the curve.

Examples.

I. On the ellipse.

(1) $y^2 - xy + \frac{5}{4}x^2 - 2y + 3x - 14 = 0.$

This ellipse has the form assigned to it in fig 66; A being the origin of the axes, and C the centre.

(2) $y^2 - 2xy + 2x^2 - 2x = 0.$

The ellipse, in this case, cuts the axis of x, and touches that of y.

(3) $$y^2+xy+x^2-4y-5x-2=0;$$
the inclination of the axes being supposed $=60^\circ$.

The curve is, in this case, a circle.

(4) $$y^2-4xy+5x^2-2y+5=0.$$

This equation may be put under the form
$$\{y-(2x+1)\}^2+(x-2)^2=0,$$
and, therefore, represents a point.

II. On the hyperbola.

(1) $$3x^2-4y^2-4xy+8y+8x-12=0.$$

The co-ordinates of the centre are $x=-\frac{1}{2}$, $y=\frac{5}{4}$.

The equations to the asymptotes are
$$y=1-\tfrac{1}{2}x\pm(x+\tfrac{1}{2}).$$

(2) $$y^2-2xy-2=0.$$

In this case, the origin is at the centre, and one of the asymptotes is parallel to the axis of x.

(3) $$xy+2y+3x-1=0.$$

Here the curve is referred to axes parallel to the asymptotes.

(4) $$y^2-2xy-2y+4x-1=0.$$

The curve does not meet the diameter $y=x+1$.

The equations to the asymptotes are
$$y=2 \text{ and } y=2x.$$

III. On the parabola.

(1) $$y^2+x^2-2xy-4y+3x+3=0,$$
$$\text{or } (y-x)^2-4y+3x+3=0.$$

If an ordinate be drawn at a distance $=1$, to the *left* of the origin, it will touch the parabola, and limit the curve in that direction.

(2) $$(y+x)^2 - by + 9 = 0.$$

The parabola of which this is the equation, extends indefinitely to the *left* of the origin, and is limited towards the right by the axis of y.

(3) $$(y-x)^2 + 2\,(y-x) - 3 = 0.$$

This equation being $= (y-x+3)\,(y-x-1)$

the curve is reduced to two parallel straight lines.

(4) $$(y-x)^2 + 6\,(y-x) + 9 = 0.$$

As the equation may be put under the form

$$(y-x+3)^2$$

the curve is in this case reduced to a single straight line.

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CHAP. X.

ON THE CONSTRUCTION OF CUBIC AND BIQUADRATIC EQUATIONS.

214. It was proved in the INTRODUCTION, that the root of a simple equation, and the roots of a quadratic, might be found geometrically, the former by the intersection of two straight lines, the latter by the intersection of a straight line and circle. The principal aim of this Chapter is to shew, that equations of the third and fourth degree may, in like manner, be determined by the intersection of lines of the second order.

The method to be employed for this purpose, is founded on the following principle, that "any indeterminate equation of which x is the unknown quantity, may be conceived to result from eliminating another unknown quantity y, between two indeterminate equations which involve both x and y." The loci of these equations being then traced, the abscissas of the points in which they intersect each other will be the roots of the proposed equation.

215. With the view of rendering this process intelligible, we shall apply it to the case of a quadratic equation.

$$\text{Let } x^2 - 2px + q = 0 \text{ .... } (1).$$

$$\text{Assume } x^2 = ky \text{ .... } (2).$$

k being any constant quantity.

Therefore equation (1) becomes, by substitution,

$$ky - 2px + q = 0 \text{ .... } (3).$$

Hence, the proposed equation may be considered as resulting from the elimination of y between (2) and (3). The roots of (1), therefore, will be the abscissas of the intersection of the loci represented by the last two equations.

Since k may be taken at pleasure, we shall assume it $=2p$; let AX, AY, be two rectangular axes, ZAz a parabola, whose principal parameter is $2p$, and whose axis is in the direction AY, then, if the straight line whose equation is (3) be drawn, it will cut the parabola in two points, of which the abscissas will be the roots required.

We have supposed the line (3) to cut the parabola in *two* points, thus implying that the roots of the given equation are real and unequal. If the roots be equal, the line will touch the parabola, and if they be imaginary, it will not meet the parabola at all.

216. PROP. 1. To construct the roots of a cubic equation, whose second term is wanting.

Let the proposed equation be

$$x^3 \pm qx + r = 0,$$

or, when made homogeneous,

$$x^3 \pm \lambda . qx + \lambda^2 . r = 0.$$

Multiplying each term by x, we have

$$x^4 \pm \lambda . qx^2 + \lambda^2 rx = 0.$$

(1) Let the coefficient of x^2 be supposed positive, or

$$x^4 + \lambda . qx^2 + \lambda^2 rx = 0.$$

$$\text{Assume } x^2 = ky \ldots\ldots\ldots (1),$$

k being any constant quantity; then, by substitution, the equation becomes

$$k^2y^2 + \lambda qky + \lambda^2 rx = 0,$$

$$\text{or } y^2 + \frac{\lambda q}{k} y + \frac{\lambda^2 r}{k^2} x = 0,$$

$$\text{but } x^2 - ky = 0,$$

adding therefore these two equations,

$$y^2 + x^2 + \frac{\lambda q - k^2}{k} y + \frac{\lambda^2 r}{k^2} x = 0.$$

Now as k^2 is quite arbitrary, we may assume it $= \lambda q$, whence the equation just obtained becomes

$$y^2 + x^2 + \lambda \cdot \frac{r}{q} x = 0 \text{ ...... } (2),$$

which is the equation to a circle, passing through the origin, whose centre is on the axis of x, and whose radius $= \lambda \dfrac{r}{2q}$.

Hence the roots required are thus found.

Let AX, AY be the rectangular axes, (fig. 71.) describe a parabola ZAz, whose vertex is A, and latus-rectum $= \sqrt{(\lambda q)}$; to the *left* of A assume $AC = \dfrac{\lambda r}{2q}$, and upon it describe a circle, cutting the parabola in the point P; then the ordinate PM being drawn, AM will be the root required.

At the origin A, the circle and parabola again intersect each other, and $x = 0$; which is, therefore, the root that was introduced by multiplying each term by x.

In this case, the remaining roots will be imaginary, and the real root will be positive or negative, according as the sign of the last term (r) is negative or positive.

The case just considered is that which is reducible by Cardan's Rule.

(2) Let the coefficient of the second term be negative, or

$$x^4 - \lambda q x^2 + \lambda^2 r x = 0.$$

Assume, as before, $x^2 = \sqrt{\lambda q}\,.\,y \ldots\ldots (1)$;

therefore, by substitution, the equation becomes,

$$\lambda q\,.\,y^2 - \lambda q\,.\,x^2 + \lambda^2 r x = 0,$$

or dividing by λq,

$$y^2 - x^2 + \frac{\lambda r}{q}\,x = 0 \ldots\ldots\ldots (2).$$

Which is the equation to an equilateral hyperbola, whose vertex is the origin, and whose axis coincides with AX.

Let ZAz (fig. 72.) be, as before, the parabola which is the locus of equation (1). To the *left* of A, assume $AC = \frac{\lambda x}{q}$, and on AC as axis, describe an equilateral hyperbola, one branch of which will cut the parabola in two points P', P'', and the other passing through A, will cut the parabola only in one point P; hence the roots are all *real* in this case, and are AM, AM', AM''. Two of them will be positive or negative, according as the last term is negative or positive.

The point A, as before, corresponds to $x = 0$.

If the hyperbola *touch* the parabola, two of the roots will be equal, and if it do not meet the parabola, two roots will be imaginary.

217. Cor. A cubic equation which has all its terms, may be constructed on the same principles.

For instance, let the equation, when made homogeneous, be

$$x^3 + px^2 + \lambda\,.\,qx + \lambda^2 r = 0.$$

$$\text{Assume } x^2 = ky \ldots\ldots\ldots\ldots (1);$$

$$\therefore\ kxy + pky + \lambda qx + \lambda^2 r = 0,$$

$$\text{or if } k = p,\ xy + py + \frac{\lambda q}{p}x + \frac{\lambda^2 r}{p} \ldots\ldots (2),$$

which is the equation to an hyperbola whose asymptotes are parallel to the axes.

The loci, therefore, of (1) and (2) being constructed, their intersections will determine the roots.

218. **Prop. 2.** To construct the roots of a biquadratic equation, whose second term is wanting.

Let the equation be $x^4 + qx^2 + rx + s = 0$,

which, when made homogeneous, becomes

$$x^4 + \lambda . qx^2 + \lambda^2 . rx + \lambda^3 . s = 0.$$

(1) Let the coefficient of x^2 be supposed positive.

$$\text{Assume } x^2 = ky \text{ ...... } (1);$$

therefore, by substitution, we have

$$k^2y^2 + \lambda qx^2 + \lambda^2 rx + \lambda^3 s = 0.$$

Let k^2 be taken $= \lambda q$, then, dividing each term by λq,

$$y^2 + x^2 + \lambda \frac{r}{q} x + \frac{\lambda^2 s}{q} = 0 \text{ ...... } (2).$$

which is the equation to a circle whose centre is on the axis of x, at a distance $\frac{\lambda r}{2q}$ to the *left* of the origin, and whose radius $= \frac{\lambda}{2q} \sqrt{(r^2 - 4qs)}$.

Hence, the roots may be thus found.

Describe, as in the last proposition, the parabola ZAz: to the left of A take $AC = \frac{\lambda r}{2q}$, and from C as centre with radius $= \frac{\lambda}{2q} \sqrt{(r^2 - 4qs)}$ describe a circle cutting the parabola in the points P, P', then, the abscissas AM, AM' are the roots required, both of which are negative.

If the radius of the circle be greater than AC, the circle cuts the other branch of the parabola, and therefore one of the roots will be positive, and the other negative.

This takes place when $\frac{\lambda}{q}\sqrt{(r^2 - 4qs)} > \frac{\lambda r}{q}$,

$$\text{or } r^2 - 4qs > r^2,$$

$$\text{or } -4qs > 0;$$

but as (q) is, by supposition, positive, this condition cannot obtain unless (s) be negative: hence, when the last term of the equation is negative, one of the roots is positive, and the other negative.

Since the circle cannot cut the parabola in more than two points, the remaining roots must be imaginary.

If the circle touch the parabola, two of the roots are equal, and if it do not meet the parabola, all the roots are imaginary.

(2) Let the coefficient of x^2 be negative.

Then, making the same assumption as before, the proposed equation becomes

$$y^2 - x^2 + \frac{\lambda n}{q}x + \frac{\lambda^2 s}{q} = 0,$$

which is the equation to an equilateral hyperbola, whose centre is on the axis of (x) at a distance $\frac{\lambda r}{2q}$ from the origin, and whose axis $= \frac{\lambda}{2q}\sqrt{(r^2 - 4qs)}$.

Let $PP'l$, $pp'l$, fig. 73, be the branches of this hyperbola, and let it intersect the parabola ZAz in the points P, P'; p, p';

then the corresponding abscissas AM, AM', Am, Am', are the roots required.

Two of these roots will be positive and two negative, if the last term be negative.

Two of them will be equal, or imaginary, according as either branch of the hyperbola touches or does not meet the parabola; and they will all be imaginary, if neither branch of the hyperbola meet the parabola.

219. **Cor.** A biquadratic which has all its terms, may, in like manner, be constructed.

$$\text{Let } x^4 + px^3 + \lambda q x^2 + \lambda^2 r x + \lambda^3 s = 0.$$

$$\text{Assume } x^2 = ky \ldots\ldots\ldots\ldots (1);$$

$$\therefore\ k^2y^2 + pkxy + \lambda q ky + \lambda^2 r x + \lambda^3 s = 0 \ldots\ldots (2),$$

which is the equation to an hyperbola, one of whose asymptotes is parallel to the axis of (x).

The curves (1) and (2) being drawn, the abscissas of the points in which they intersect are the roots required.

220. **Prop. 3.** To trisect a given angle.

Let θ be the given angle; then, by Trigonometry,

$$\cos\theta = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}.$$

$$\text{Assume } \cos\frac{\theta}{3} = x;$$

$$\therefore\ \cos\theta = 4x^3 - 3x;$$

therefore, transposing, and making the equation homogeneous by introducing a radius (r),

$$4x^3 - 3r^2x - r^2\cos\theta = 0;$$

$$\text{or } x^3 - \frac{3}{4}r^2x - \frac{r^2}{4}\cos\theta = 0.$$

In order to construct the roots of this equation, we must multiply each term by x;

$$\therefore\ x^4 - \frac{3}{4} r^2 x^2 - \frac{r^2}{4} \cos\theta \cdot x = 0.$$

$$\text{Assume } x^2 = \frac{r}{2} \sqrt{3} \cdot y \ \ldots\ldots\ (1);$$

therefore, by substitution and reduction, we have

$$y^2 - x^2 - \tfrac{1}{3} \cos\theta \cdot x = 0 \ \ldots\ldots\ (2),$$

which is the equation to an equilateral hyperbola, whose vertex is the origin, and whose major axis coincides with AX.

Let ZAz, (fig. 74), be the parabola which is the locus of (1). To the right of A take $AV = \frac{1}{3}\cos\theta$, and on AV as a diameter describe an equilateral hyperbola, one branch of which will cut the parabola in two points P, P', and the other passing through A will cut the parabola only in one point P''. The ordinates PM, $P'M'$, $P''M''$ being drawn, AM, AM', AM'' will be roots required.

The three analytical values of x are

$$\cos\frac{\theta}{3},\quad \cos\frac{2\pi - \theta}{3},\quad \text{and } \cos\frac{2\pi + \theta}{3}.$$

See *Woodhouse's Trigonometry*, Chap. vii.

221. Prop. 4. To find a cube that shall be double of a given cube.

Let (a) be the side of the given cube, x that of the required cube, then we have

$$x^3 = 2a^3;$$

$$\therefore\ x^4 - 3a^3 x = 0.$$

$$\text{Assume } x^2 = ay \ \ldots\ldots\ (1);$$

therefore, by substitution,

$$a^2y^2-2a^3x=0,$$

$$\text{or } y^2=2ax \; \ldots\ldots \; (2).$$

On *AY*, (fig. 75), as axis describe a parabola whose latus rectum is (a), and on *AX* as axis, another parabola whose latus rectum is $2a$: let the parabolas intersect each other in *P*, then the ordinate *PM* being drawn, *AM* will be the side of the cube required.

222. Cor. In like manner, a cube may be found which is (m) times a given cube.

223. Prop. 5. To find two mean proportionals between two given lines (a) and (b).

Let x and y be the mean proportionals sought, then

$$\frac{a}{x}=\frac{x}{y}=\frac{y}{b};$$

$$\therefore \; ay=x^2 \; \ldots\ldots\ldots \; (1),$$

$$\text{and } bx=y^2 \; \ldots\ldots\ldots \; (2).$$

On *AY*, *AX* respectively, as axes, construct two parabolas whose principal parameters are (a) and (b). Let the parabolas intersect in *P*, and draw the ordinate *PM*. Then, *AM*, *MP* are the mean proportionals required.

PART II.

ANALYTICAL GEOMETRY OF THREE DIMENSIONS.

SECTION I.

ON THE STRAIGHT LINE AND PLANE.

CHAP. I.

ON THE POSITION OF A POINT IN SPACE.

224. THE method of determining the position of any number of points in space, is analogous to that which was employed in Art. 44, to fix the position of a system of points situated in one plane.

Let $f(x, y, z) = 0$ be any indeterminate equation between three variables; then, the numerical value of any one of them can be determined only by assigning arbitrary values to the other two. In order to represent geometrically the infinite number of solutions which may be thus obtained, we proceed as follows.

Suppose XAY, XAZ, YAZ, (fig. 76), to be three planes, whose mutual inclination is known, and let AX, AY, AZ, be the lines in which each two of these planes intersect one another. In AX, take AM equal to any assumed value of x, draw MN parallel to AY, and equal to any assumed value of y, then, if NP be drawn parallel to AZ, and equal to the resulting value of z, each point P, so determined, will furnish a solution of the equation $f(x, y, z) = 0$.

The surface, plane or curved, in which all the points P are situated, is called the *locus* of the equation $f(x, y, z) = 0$.

225. The position, therefore, of the point P will thus be determined, when the lines AM, MN, NP are known. Now, if the parallelepiped AP be completed, it is manifest that $AM = Pm =$ the distance of P from the plane YAZ, measured in the direction AX; and that MN, PN, are respectively equal to P's distance from the planes XAZ, XAY, measured in the directions AY, AZ. It thence follows that *the position of a point in space depends on its distances from three given planes, estimated in the direction of the lines in which they intersect.*

The lines AM, MN, NP, are called the *co-ordinates* of the point P, and A is said to be the *origin*, AX, AY, AZ the *axes*, of the three *co-ordinate planes* XAY, XAZ, YAZ. We commonly denote AM by x, MN by y, NP by z; whence these lines are called respectively the axes of x, of y, and of z: also, XAY is, for the sake of brevity, called the plane of xy, XAZ the plane of xz, and YAZ the plane of yz.

The equation which expresses the relation between the co-ordinates of any point of a surface is called the *equation* to that surface. Thus, if $AM = a$, $MN = b$, $NP = c$, then, $x = a$, $y = b$, $z = c$, is the equation to the point P.

226. The three co-ordinate planes, when indefinitely produced form eight trihedral* angles at the point A, of which four are situated above, and four beneath, the horizontal plane xy. Now, since the co-ordinates x, y, z, are measured from a fixt point A along lines given in position, the general rule (Schol. p. 35.) for the interpretation of the signs will enable us to determine in which of those eight angles the point P is situated.

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* A solid angle is denominated *dihedral*, *trihedral*, &c. according as it is contained by *two*, *three*, &c. planes.

Thus, let a, b, c be the absolute values of the co-ordinates of any point P, *above* the plane of xy; then if

$x = +a$, $y = +b$, $z = +c$, the point is in the angle $AXYZ$,

$x = +a$, $y = -b$, $z = +c$ . . . . $AXY'Z$,

$x = -a$, $y = +b$, $z = +c$ . . . . $AX'YZ$,

$x = -a$, $y = -b$, $z = +c$ . . . . $AX'Y'Z$.

By changing the signs of the co-ordinates in each of these four lines, the position of the point P, when symmetrically situated *beneath* the plane of xy, will be determined.

Hence,

(1) When P is in the plane of xy, its distance z from this plane $= 0$; therefore the equation to P is

$$x = a, \quad y = b, \quad z = 0.$$

In like manner, the equation to P, when it is situated in the planes of xz, yz respectively, are

$$x = a, \quad z = c, \quad y = 0,$$

$$y = b, \quad z = c, \quad x = 0.$$

(2) When P is on the axis of

x, its equation is $x = a$, $y = 0$, $z = 0$,

y, ... $y = b$, $x = 0$, $z = 0$,

z, ... $z = c$, $x = 0$, $y = 0$.

(3) When P coincides with the origin, its equation is

$$x = 0, \quad y = 0, \quad z = 0.$$

227. The points N, n, m, in which the lines PN, Pn, Pm, meet the planes of xy, of xz and of yz, are called the *projections* of the point P upon these respective planes.

Hence, by the last article,

$x=a$, $y=b$ is the equation to the projection of P upon xy,
$x=a$, $z=c$ . . . . xz,
$y=b$, $z=c$ . . . . yz.

It is evident, that if any two of these equations be given, the third is necessarily known: whence it follows, that *the position of a point in space is determined, when its projections on any two of the co-ordinate planes, are known.*

228. The mutual inclination of the co-ordinate planes being altogether arbitrary, we shall suppose, for the sake of simplicity, that each of them is at right angles to the other two. The planes, as well as the co-ordinates, are in this case said to be *rectangular*.

We shall always consider the plane of xy as *horizontal*, and the planes of xz, yz, as *vertical*.

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CHAP. II.

ON THE STRAIGHT LINE IN SPACE.

229. **Def.** If through a given straight line in space a plane be drawn perpendicularly to any of the co-ordinate planes, the intersection of these planes will be the *projection* of the given line. The plane thus drawn, is called the *projecting plane.*

We generally suppose the straight line to be projected on the planes of xz and yz; if, therefore, their intersection AZ be considered as the axis of the abscissas, then AX, AY will be the axes of the ordinates in the planes xz, yz, respectively.

230. The position of a straight line in space will be determined, when its projections on any two of the co-ordinate planes are known; for the proposed line will evidently be the common section of the two projecting planes.

Thus, let pp', qq', (fig. 77.) be the projections of the line PP' on the planes of xz, yz; then, drawing the projecting planes $PpP'p'$, $PqP'q'$, their intersection PP' will be the line required.

231. Prop. 1. To find the equations to a straight line in space.

$$\text{Let } x = az + \alpha$$

$$y = bz + \beta$$

be the equations to the projections of any straight line PP' on the planes of xz and of yz; then will they be the equations to the line itself.

For the former of these equations, being independent of y, is the equation not to pp' only, but to every line in the projecting plane $PpP'p'$. The latter equation is, in like manner, the equation to every line in the projecting plane $PqP'q'$. Therefore the system of equations

$$\left.\begin{aligned} x &= az + \alpha \\ y &= bz + \beta \end{aligned}\right\} \ldots\ldots (1)$$

being common to the two projecting planes, must belong to PP', which is the line of their intersection.

It is to be observed, that a, b denote the tangents of the inclination of pp', qq' to AZ, and that α, β express the distances from the origin to the points in which the same lines intersect AX, AY.

232. Cor. 1. By eliminating z from the foregoing equations, we have

$$y - \beta = \frac{b}{a}(x - \alpha) \ldots\ldots (2)$$

which is the equation to the projection of PP' on the plane of xy.

233. Cor. 2. The point in which the line PP' meets the planes of xy, of xz, or of yz, is found by making z, y or x respectively $= 0$, in equations (1) and (2).

234. Cor. 3. When PP' passes through the origin, its equations are

$$x = az, \quad y = bz.$$

235. Prop. 2. To find the equations to a straight line, passing through a given point (x', y', z').

Since the given co-ordinates must satisfy the general equations to a straight line

$$\left.\begin{aligned} x &= az + \alpha \\ y &= bz + \beta \end{aligned}\right\} \ldots\ldots (1)$$

We therefore have

$$x' = az' + a, \text{ and } y' = bz' + \beta;$$

$$\therefore\ a = x' - az', \text{ and } \beta = y' - bz'.$$

These values of a and β being substituted in (1), there result

$$\left.\begin{aligned} x - x' &= a(z - z') \\ y - y' &= b(z - z') \end{aligned}\right\} \ldots\ldots (2)$$

which are the equations required.

236. Cor. Hence may be found the equations to a line passing through *two* given points.

Let x'', y'', z'' be the co-ordinates of the second point.

Substituting these quantities for x, y, z, in (2), we have

$$a = \frac{x'' - x'}{z'' - z}, \quad b = \frac{y'' - y'}{z'' - z'};$$

$$\therefore\ x - x' = \frac{x'' - x'}{z'' - z'}(z - z'),$$

$$y - y' = \frac{y'' - y'}{z'' - z'}(z - z'),$$

are the equations sought.

237. Prop. 3. To find the conditions requisite for the intersection of two straight lines, and also the co-ordinates of the point in which they do intersect.

Let the equations to the lines be

$$x = az + a; \quad y = bz + \beta.$$

$$\text{and } x = a'z + a'; \quad y = b'z + \beta'.$$

Then, supposing them to intersect, the co-ordinates at the point of their intersection will be identical; therefore, subtracting the lower from the upper equations,

$$(a-a')\,z+\alpha-\alpha'=0,$$
$$(b-b')\,z+\beta-\beta'=0;$$

therefore, eliminating z, we have

$$\frac{\alpha-\alpha'}{a-a'}=\frac{\beta-\beta'}{b-b'},$$

or $(\alpha-\alpha')\,(b-b')=(\beta-\beta')\,(a-a') \; \ldots\ldots \; (1)$

which is the equation of condition required.

Next to find the co-ordinates at the point of intersection.

Since $z=\dfrac{\alpha'-\alpha}{a-a'}$ we immediately deduce,

$$x=\frac{a\alpha'-a'\alpha}{a-a'}$$

and $y=\dfrac{b\beta'-b'\beta}{a-a'}$

238. **Cor.** When the lines are parallel, the co-ordinates of the point of intersection are infinitely great;

$$\therefore\; a=a', \text{ and } b=b'.$$

239. **Prop. 4.** To find the distance of any point from the origin.

Let P, (fig. 78.) be the given point, AM, MN, NP its co-ordinates. Then in the right-angled triangle ANP,

$$AP^2=AN^2+NP^2,$$

and in the right-angled triangle AMN,

$$AN^2=AM^2+MN^2;$$
$$\therefore\; AP^2=AM^2+MN^2+NP^2,$$

or if $AP=\rho$,

$$\rho^2=x^2+y^2+z^2.$$

Hence, in a right-angled parallelepiped, the square of the diagonal is equal to the sum of the squares of the three edges.

240. Cor. Since $x=az$, and $y=bz$, we have, by substitution,

$$\rho^2=z^2\{1+a^2+b^2\};$$

$$\therefore\ z=\frac{\pm\rho}{\sqrt{(1+a^2+b^2)}}.\quad \text{Hence,}$$

$$x=\frac{\pm a\rho}{\sqrt{(1+a^2+b^2)}},$$

$$y=\frac{\pm b\rho}{\sqrt{(1+a^2+b^2)}}.$$

241. Prop. 5. To find the distance between *two* points.

Let P, P' be the two points, and x', y', z' the co-ordinates of the latter.

Then PP' may be considered as the diagonal of a parallelepiped, whose three edges are $x'-x$, $y'-y$, $z'-z'$; we therefore have, by the last Prop.

$$PP'^2=(x'-x)^2+(y'-y)^2+(z'-z)^2.$$

242. Cor. Let $PP'=\delta$, $AP'=\rho'$.

Then, expanding the value of PP'^2, and reducing it by means of (239), we have

$$\delta^2=\rho^2+\rho'^2-2\{xx'+yy'+zz'\}.$$

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CHAP. III.

ON THE INCLINATION OF STRAIGHT LINES TO THE AXES, AND TO ONE ANOTHER.

243. **Prop. 1.** To find the inclination of a straight line to each of the axes.

Let AP be the given line (ρ); then, (fig. 78.), retaining the same notation, we have

$$x = AP \cos PAX = \rho \cos \rho, x;$$

$$\therefore \cos \rho, x = \frac{x}{\rho} = \frac{\pm a}{\sqrt{(1+a^2+b^2)}} \quad (240).$$

In like manner,

$$\cos \rho, y = \frac{\pm b}{\sqrt{(1+a^2+b^2)}}$$

$$\cos \rho, z = \frac{\pm 1}{\sqrt{(1+a^2+b^2)}}$$

The two angles which ρ forms with each of the axes are supplements of one another; of these, the acute corresponds to the positive, and the obtuse to the negative, sign.

244. **Cor. 1.** Squaring these values, and adding the results, we have

$$\cos^2 \rho, x + \cos^2 \rho, y + \cos^2 \rho, z = 1, \textit{ to radius } 1 \ldots. (1),$$

$$= r^2, \textit{ to radius } r \ldots. (2).$$

245. Cor. 2. If the symbols ρ, xy; ρ, xz; ρ, yz, denote the angles which ρ makes with the planes of xy, xz, yz respectively, we shall have

$$\sin^2 \rho, yz + \sin^2 \rho, xz + \sin^2 \rho, xy = 1 \ldots . (3).$$

The equations marked (1) and (3), express the relations subsisting among the angles which a straight line in space forms with the three axes, and with the three co-ordinate planes. Whence it is evident, that if any two of these angles be given, the third is known.

246. Prop. 2. The separate inclinations of two lines in space to the axes being given, to find their mutual inclination.

Since two lines in space do not necessarily intersect each other, draw through the origin two lines respectively parallel to the given lines. In these take any points P, P', join PP', and let $AP = \rho$, $AP' = \rho'$.

$$\text{Then (242) } PP'^2 = \rho^2 + \rho'^2 - 2\{xx' + yy' + zz'\},$$

$$\text{and also} = \rho^2 + \rho'^2 - 2\rho\rho' \cos \rho, \rho';$$

$$\therefore \rho\rho' \cos \rho, \rho' = xx' + yy' + zz' \ldots\ldots\ldots\ldots (1).$$

But (242) $x = \rho \cos \rho, x$; $y = \rho \cos \rho, y$; $z = \rho \cos \rho, z$.

Similarly, $x' = \rho' \cos \rho', x$; $y' = \rho' \cos \rho', y$; $z' = \rho' \cos \rho', z$;

therefore substituting in (1), and dividing by $\rho\rho'$, we have

$$\cos \rho, \rho' = \cos \rho, x \cos \rho', x + \cos \rho, y \cos \rho', y + \cos \rho, z \cos \rho', z.$$

247. Cor. 1. When the lines are at right angles,

$$\cos \rho, x \cos \rho', x + \cos \rho, y \cos \rho', y + \cos \rho, z \cos \rho', z = 0,$$

and when they are parallel,

$$\cos \rho, x \cos \rho', x + \cos \rho, y \cos \rho', y + \cos \rho, z \cos \rho', z = 1.$$

248. Cor. 2. If $\cos \rho, x = m$; $\cos \rho, y = n$; $\cos \rho, z = p$;

$\cos \rho', x = m'$; $\cos \rho', y = n'$; $\cos \rho', z = p'$;

Then $\cos \rho, \rho' = \quad mm' + nn' + pp' \ldots\ldots$ *to rad.* 1 .. (1),

$$\text{or } (244) = \pm \frac{mm' + nn' + pp'}{\sqrt{(m^2 + n^2 + p^2)(m'^2 + n'^2 + p'^2)}} \text{ to rad. } r \ldots (2).$$

249. Prop. 3. To find the equations to a straight line, in its terms of the angles which it forms with the three axes.

The general equations are

$$x = az + a, \text{ and } y = bz + \beta \ldots\ldots (1).$$

If a line (ρ) parallel to this line be drawn through the origin, its equations will be

$$x = az, \text{ and } y = bz;$$

$$\therefore \frac{x}{z} = a, \quad \text{and } \frac{y}{z} = b, \ldots\ldots (2).$$

Now $x = \rho \cos \rho, x$; $y = \rho \cos \rho, y$, and $z = \rho \cos \rho, z$;

$$\therefore \rho = \frac{x}{\cos \rho, x} = \frac{y}{\cos \rho, y} = \frac{z}{\cos \rho, z};$$

$$\therefore \frac{x}{z} = \frac{\cos \rho, x}{\cos \rho, z} = \therefore a$$

$$\frac{y}{z} = \frac{\cos \rho, y}{\cos \rho, z} = \therefore b$$

Substituting these values of a and b in (1), we have

$$x = \frac{\cos \rho, x}{\cos \rho, z} z + a$$

$$y = \frac{\cos \rho, y}{\cos \rho, z} z + \beta$$

which are the equations required.

250. **Cor. 1.** According to the notation employed above, (248), these equations are

$$x = \frac{m}{p} z + \alpha,$$

$$y = \frac{n}{p} z + \beta.$$

251. **Cor. 2.** Since $m = ap$; $n = bp$;

and similarly, $m' = a'p'$; $n' = b'p'$,

we have, by substitution, in (248),

$$\cos \rho, \rho' = \pm \frac{1 + aa' + bb'}{\sqrt{(1 + a^2 + b^2)(1 + a'^2 + b'^2)}},$$

which expresses the inclination of two lines, whose equations are

$$x = az + \alpha,$$

$$\text{and } y = bz + \beta.$$

CHAP. IV.

ON THE PLANE SITUATED IN SPACE.

252. A PLANE is generated by a straight line which moves in a direction parallel to itself, along a straight line given in position.

Of these straight lines, the former is called the *generating line*, the latter, the *directrix*.

PROP. 1. To find the equation to a plane.

Let the equations to the generating line be

$$x = az + \alpha, \text{ and } y = bz + \beta \ldots\ldots\ldots\ldots (1),$$

and the equation to the directrix

$$Y = mX + n \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

it being supposed to lie in the plane of xy.

The equation sought will be obtained by expressing analytically that the generating line remains parallel to itself while it moves through each point of the directrix.

The generating line, when in any position, will be defined by the equations

$$x = az + \alpha', \quad y = bz + \beta'.$$

But, since it passes by hypothesis through a point

$$x = X, \quad y = Y, \quad z = 0,$$

of the directrix, we have

$$X = \alpha' = x - az,$$
$$Y = \beta' = y - bz;$$

therefore, substituting these values of X and Y in equation (2), we have

$$y - bz = m(x - az) + n,$$

$$\text{or } y - mx + (ma - b)z - n = 0,$$

$$x - \frac{1}{m}y + \frac{b - ma}{m}z + \frac{n}{m} = 0,$$

the equation required.

In order to give a more symmetrical form to this equation, assume

$$\frac{B}{A} = -\frac{1}{m}, \quad \frac{C}{A} = \frac{b - ma}{m}, \quad \text{and } \frac{D}{A} = \frac{n}{m};$$

then, the equation to the plane will become

$$Ax + By + Cz + D = 0.$$

253. Cor. 1. When the plane passes through the origin, $D = 0$; therefore, the equation becomes

$$Ax + By + Cz = 0.$$

254. Cor. 2. Since $z = -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}$,

$$\text{if } -\frac{A}{C} = A', \quad -\frac{B}{C} = B', \quad -\frac{D}{C} = C', \text{ then}$$

$$z = A'x + B'y + C',$$

under which form it is sometimes convenient to consider the equation to the plane.

It thence appears that *three* constants only are necessary to determine the position of a plane.

255. (1) If the plane meet AZ, then, since x and y each $= 0$, $z = -\frac{D}{C}$.

Similarly, according as the plane meets AX or AY,

$$x = -\frac{D}{A}, \quad y = -\frac{D}{B}.$$

(2) If the plane be perpendicular to AZ, then, since each point of the given plane is equidistant from that of xy, z must be equal to a constant quantity.

Similarly, x and y will be constant quantities, according as the plane is perpendicular to AX or AY.

(3) If the plane be parallel to AZ, then, $\frac{D}{C}$ being infinitely great, $C = 0$.

Similarly, $A = 0$, or $B = 0$, according as the plane is parallel to AX or to AY.

256. (1) If the plane meet the plane of xy, then $z = 0$, and the equation to their intersection is

$$Ax + By + D = 0.$$

Similarly, according as the plane meets the planes of xz, or of yz, the equation to their intersection is

$$Ax + Cz + D = 0,$$

$$\text{or } By + Cz + D = 0.$$

Note. These intersections are called the *traces* of the given plane.

(2) If the plane be perpendicular to that of xy, then, since it must be parallel to AZ, $C=0$; therefore the equation to the *trace* is

$$Ax+By+D=0.$$

The same reasoning being applicable to the remaining two co-ordinate planes, we conclude that *when a plane is perpendicular to any one of the co-ordinate planes, its equation is that of its trace upon the same plane.*

(3) If the plane be parallel to that of xy, then the co-ordinates of its intersection with x and y will be infinitely great. Hence, $-\frac{D}{A}$ and $-\frac{D}{B}$ being infinite, $A=0$ and $B=0$; therefore the equation becomes $Cz+D=0$.

Similarly, according as the plane is parallel to that of xz or of yz, $By+D=0$, or $Ax+D=0$.

257. PROP. 2. The points in which a plane intersects the axes being given, it is required to find its equation.

Let the plane meet AX, AY, AZ in the points B, C, D, fig. 79, and assume

$$AB \text{ or } -\frac{D}{A}=a,$$

$$AC \text{ or } -\frac{D}{B}=b,$$

$$AD \text{ or } -\frac{D}{C}=c.$$

Then, the general equation to the plane becomes, by substitution,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \ldots\ldots\ldots (1),$$

which is the equation sought.

258. **Prop. 3.** To find the equation to a plane, in terms of the perpendicular dropped upon it from the origin, and the angles which it forms with the axes.

Let the perpendicular be denoted by ρ.

Then, the general equation to a plane being

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

it is evident that

$$a=\frac{\rho}{\cos \rho, x}, \quad b=\frac{\rho}{\cos \rho, y}, \quad c=\frac{\rho}{\cos \rho, z};$$

therefore, by substitution, we have

$$\rho=x \cos \rho, r+y \cos \rho, y+z \cos \rho, z \ldots\ldots (2),$$

which is the equation required.

Equations (1) and (2) are analogous to those of the straight line in Art. 50, and retain the same form, whatever be the inclination of the co-ordinate planes.

259. **Cor.** When the plane passes through the origin, then

$$x \cos \rho, x+y \cos \rho, y+z \cos \rho, z=0.$$

260. **Prop. 4.** Required the equation to a plane which is parallel to the given plane

$$Ax+By+Cz+D=0.$$

Let the equation sought be

$$A'x+B'y+C'z+D'=0.$$

Then, since the two planes are parallel, their traces upon each of the co-ordinate planes will also be parallel.

Now, their traces upon xy are defined by the equations

$$Ax + By + D = 0,$$

and $A'x + B'y + D' = 0$;

therefore (57. Cor. 3.)

$$\frac{A'}{B'} = \frac{A}{B}, \text{ or } \frac{A'}{A} = \frac{B'}{B}. \quad \text{Similarly, } \frac{A'}{A} = \frac{C'}{C}.$$

Assume $\frac{A'}{A} = m$, then we have, by substitution,

$$Ax + By + Cz + \frac{D'}{m} = 0,$$

the equation required.

261. Cor. Hence, through a given point (x', y', z') a plane may be drawn parallel to the plane

$$Ax + By + Cz + D = 0.$$

The equation to the plane sought, will, by the proposition, be

$$Ax + By + Cz + d = 0,$$

and, as it passes through a point x', y', z', we have

$$Ax' + By' + Cz' + d = 0;$$

therefore eliminating (d), we obtain

$$A(x - x') + B(y - y') + C(z - z') = 0,$$

which is the equation required.

CHAP. V.

ON PROBLEMS RELATING TO THE STRAIGHT LINE AND PLANE.

262. **Prob. 1.** To find the point in which a given plane is intersected by a straight line, whose equations are

$$x = az + \alpha,\ y = bz + \beta.$$

At the point sought, the co-ordinates of the line and of the plane being identical, we have, by substituting the values of x and y, in the equation to the plane,

$$A\,(az+\alpha)+B\,(bz+\beta)+Cz+D=0,$$

or $\{Aa+Bb+C\}\ z=-\{A\alpha+B\beta+D\}$;

$$\therefore z=-\frac{A\alpha+B\beta+D}{Aa+Bb+C}.$$

$$\text{Similarly, } x=\alpha-a\,.\,\frac{A\alpha+B\beta+D}{Aa+Bb+C}$$

$$=\frac{\alpha\,\{Aa+Bb+C\}-a\,\{A\alpha+B\beta+D\}}{Aa+Bb+C},$$

$$\text{and } y=\frac{\beta\,(Aa+Bb+C)-b\,(A\alpha+B\beta+D)}{Aa+Bb+C}.$$

263. Cor. When the line is parallel to the given plane, then the co-ordinates are infinitely great; and, therefore,

$$Aa + Bb + C = 0,$$

which expresses the condition of parallelism of a straight line and plane.

264. Prob. 2. To find the equation and length of the perpendicular dropped from a given point (x', y', z') upon the plane $Ax + By + Cz + D = 0$.

(1) The equations to the perpendicular required, will be of the form

$$\left.\begin{aligned} x - x' &= a(z - z') \\ y - y' &= b(z - z') \end{aligned}\right\} \dots\dots (1),$$

in which (a) and (b) are to be determined.

Now the projections of the line and of the plane, on any one of the co-ordinate planes, are perpendicular to each other; because the projecting plane of the line being perpendicular to the given plane, their intersections with any of the co-ordinate planes, that is, the projections in question, must also be perpendicular.

The given plane, then, being projected on the planes of xz and yz, the equations to its traces are

$$Ax + Cz + D = 0,$$

$$By + Cz + D = 0;$$

$$\therefore \text{(57. Cor. 3.)} \left.\begin{aligned} x &= -\frac{C}{A}z - \frac{D}{A} \\ \text{and } y &= -\frac{C}{B}z - \frac{D}{B} \end{aligned}\right\} \dots\dots\dots (2).$$

$$\text{Hence, } a = \frac{A}{C}, \text{ and } b = \frac{B}{C};$$

therefore the equations required are

$$x - x' = \frac{A}{C}(z - z'), \text{ and } y - y' = \frac{B}{C}(z - z').$$

(2) The length (p) of the perpendicular

$$= \sqrt{\{(x - x')^2 + (y - y')^2 + (z - z')^2\}}.$$

In order to find this, assume

$$Ax' + By' + Cz' + D = D'.$$

$$\text{Now } Ax + By + Cz + D = 0;$$

therefore by subtraction,

$$A(x - x') + B(y - y') + C(z - z') + D' = 0,$$

and substituting for $x - x'$, and $y - y'$ their values above obtained, we have

$$\left\{\frac{A^2}{C} + \frac{B^2}{C} + C\right\}(z - z') = -D';$$

$$\therefore z - z' = -\frac{CD'}{A^2 + B^2 + C^2},$$

$$\text{hence, } x - x' = -\frac{AD'}{A^2 + B^2 + C^2},$$

$$y - y' = -\frac{BD'}{A^2 + B^2 + C^2};$$

$$\therefore p = \pm \frac{D'}{A^2 + B^2 + C^2} \cdot \sqrt{(A^2 + B^2 + C^2)},$$

or substituting for D' its value,

$$p = \pm \frac{Ax' + By' + Cz' + D}{\sqrt{(A^2 + B^2 + C^2)}}.$$

The lower sign is to be used, when the point is situated between the given plane, and the plane of xy.

265. Cor. If the given point be the origin, then x', y', z' each $= 0$, and

$$p = \pm \frac{D}{\sqrt{(A^2 + B^2 + C^2)}}.$$

266. *Observation.* The value of (p) may also be found by a method analogous to that made use of in Art. 56.

Conceive a plane drawn through the point (x', y', z') parallel to the given plane, and drop upon it from A a perpendicular AQ, meeting the given plane in P: then, PQ will $= p$.

$$\text{Now } AP \text{ or } \rho = x \cos \rho, x + y \cos \rho, y + z \cos \rho, z,$$

$$\text{and } AQ \quad = x' \cos \rho, x + y' \cos \rho, y + z' \cos \rho, z;$$

$$\therefore\ p = x' \cos \rho, x + y' \cos \rho, y + z' \cos \rho, z - \rho,$$

which is the value required.

267. Prob. 3. Required the equation to a plane drawn through a given point (x', y', z'), perpendicular to a given line $x = az + \alpha$, $y = bz + \beta$; required, also, the distance between the given point and the point where the plane meets the line.

(1) The equation to the plane will be of the form

$$A(x - x') + B(y - y') + C(z - z') = 0,$$

and since it is perpendicular to the given line, we have, (264),

$$A = aC, \quad B = bC;$$

therefore, substituting, and dividing by C,

$$a(x - x') + b(y - y') + z - z' = 0 \; \ldots\ldots \; (1),$$

the equation required.

(2) The distance sought is found by a method similar to that employed in the last proposition; thus the equations to the given line being written as follows:

$$x - x' = a(z - z') + \alpha - x' + az',$$

$$y - y' = b(z - z') + \beta - y' + bz',$$

Substitute these values of $x - x'$, $y - y'$ in equation (1), and we have

$$(a^2 + b^2 + 1)(z - z') + a(\alpha - x' + az') + b(\beta - y' + bz') = 0;$$

$$\therefore\ z - z' = \frac{a(x' - \alpha) + b(y' - b)}{a^2 + b^2 + 1} - z' = \frac{m}{a^2 + b^2 + 1} - z',$$

if $a(x' - \alpha) + b(y' - b)$ be taken $= m$.

$$\text{Hence, } x - x' = \frac{m}{a^2 + b^2 + 1} - (x' - \alpha),$$

$$y - y' = \frac{m}{a^2 + b^2 + 1} - (y - \beta);$$

therefore distance sought

$$= \sqrt{\{(x - x')^2 + (y - y')^2 + (z - z')^2\}}$$

$$= \sqrt{\left\{(x' - \alpha)^2 + (y' - \beta)^2 + z'^2 - \frac{m^2}{a^2 + b^2 + 1}\right\}}.$$

268. Prob. 4. To find the angle which a given straight line forms with a given plane.

Let the equations to the line be

$$x = az + \alpha,\quad y = bz + \beta,$$

and the equation to the plane

$$Ax + By + Cz + D = 0.$$

The inclination of a line to a plane, is the angle contained by the line and its projection upon the plane, and is therefore equal to the complement of the angle formed by the line and a perpendicular dropped from any point of it upon the plane.

Let the equations to the perpendicular be

$$x = a'z + \alpha', \text{ and } y = b'z + \beta',$$

$$\text{then (264), } a' = \frac{A}{C}, \text{ and } b' = \frac{B}{C}.$$

Now, in general, (251)

$$\cos \rho, \rho' = \frac{1 + aa' + bb'}{\sqrt{(1 + a^2 + b^2)(1 + a'^2 + b'^2)}} \ldots\ldots (1).$$

Let ρ be the given line, and ρ' the perpendicular dropped upon the given plane, also let the symbol ρ, Π be employed to denote the angle contained by the line and plane. Then, substituting for a' and b' their values in (1), we have

$$\sin \rho, \Pi = \frac{Aa + Bb + C}{\sqrt{(1 + a^2 + b^2)(A^2 + B^2 + C^2)}}.$$

269. Cor. When the line is parallel to the plane

$$Aa + Bb + C = 0,$$

which agrees with Art. 263.

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CHAP. VI.

ON THE INCLINATION OF PLANES, TO THE CO-ORDINATE PLANES, AND TO ONE ANOTHER.

270. **Prop. 1.** To find the inclination of a given plane Π to each of the co-ordinate planes.

Let the symbols Π, xy; Π, xz; Π, yz be employed to denote the inclinations sought.

If through the origin a plane be drawn parallel to the given plane, its equation will be (260)

$$Ax + By + Cz = 0.$$

From the origin, draw, at right angles to this plane, a line (ρ), whose equations are

$$x = az, \quad y = bz;$$

$$\text{then (264) } a = \frac{A}{C}, \quad \text{and } b = \frac{B}{C}.$$

$$\text{Now, } \cos \rho, z = \frac{\pm 1}{\sqrt{(1 + a^2 + b^2)}} \quad (243),$$

$$\text{but } \cos \rho, z = \cos \Pi, xy;$$

therefore, substituting for a and b their values,

$$\cos \Pi, xy = \frac{\pm C}{\sqrt{(A^2 + B^2 + C^2)}}.$$

$$\text{Similarly,}\quad \cos \Pi, xz = \frac{\pm B}{\sqrt{(A^2+B^2+C^2)}},$$

$$\text{and}\quad \cos \Pi, yz = \frac{\pm A}{\sqrt{(A^2+B^2+C^2)}}.$$

271. Cor. Since $\cos^2 \rho, x + \cos^2 \rho, y + \cos^2 \rho, z = 1$;

$$\therefore\ \cos^2 \Pi, yz + \cos^2 \Pi, xz + \cos^2 \Pi, xy = 1.$$

272. Prop. 2. *The separate inclinations of two planes Π and Π' to the co-ordinate planes being given, to find their mutual inclination.*

Draw through the origin two planes parallel respectively to the given planes; then, if two lines ρ and ρ', be drawn from the origin at right angles to the planes, their inclination Π, Π', will $= \angle\, \rho, \rho'$, and their equations will be (259),

$$x \cos \rho, x + y \cos \rho, y + z \cos \rho, z = 0,$$

$$x \cos \rho' x + y \cos \rho', y + z \cos \rho', z = 0.$$

Now, in general,

$$\cos \rho, \rho' = \cos \rho, x \cos \rho', x + \cos \rho, y \cos \rho', y + \cos \rho, z \cos \rho', z$$

but $\cos \rho, x = \cos \Pi, yz$; and so with respect to the others, therefore, making the requisite substitutions, we directly obtain

$$\cos \Pi, \Pi' = \cos \Pi, yz \cos \Pi', zy + \cos \Pi, xz \cos \Pi', xz + \cos \Pi, xy \cos \Pi', xy,$$

which is the inclination sought.

273. Cor. 1. When the planes are at right angles to one another,

$$\cos \Pi, zy \cos \Pi', zy + \cos \Pi, xz \cos \Pi', xz + \cos \Pi, xy \cos \Pi', xy = 0.$$

274. Cor. 2. If $\cos \rho, x = M,\ \cos \rho, y = N,\ \cos \rho, z = P,$

$$\cos \rho', x = M',\ \cos \rho', y = N',\ \cos \rho', z = P',$$

$$\text{then } \cos \Pi, \Pi' = MM' + NN' + PP' \textit{ to rad. } 1 \ldots\ldots (1),$$

$$\text{or } = \pm \frac{MM' + NN' + PP'}{\sqrt{(M^2+N^2+P^2)(M'^2+N'^2+P'^2)}} \textit{ to rad. } r \ldots . (2).$$

which two expressions are analogous to those in Art. 248.

CHAP. VII.

ON THE ORTHOGONAL PROJECTION OF PLANE FIGURES.

275. Def. 1. A system of points is said to be *orthogonally* projected upon any plane, when each point is projected by means of a perpendicular dropped upon that plane.

The projection of a line, whether straight or curved, upon any plane, may be considered as the locus of all the projected points of that line.

276. Def. 2. If the sides of any plane figure be projected orthogonally upon a plane, the figure formed by these projections is called the *orthogonal projection* of the given figure upon the plane.

As an illustration of this definition, conceive a right prism to be cut obliquely by a plane, then the base of the prism will be the projection of the resulting figure upon the plane of the base.

277. Prop. 1. The square of any plane figure is equal to the sum of the squares of its projections on three co-ordinate rectangular planes.

Let the given figure or area be denoted by A, and its projections on the planes of xy, xz, and yz, by A_z, A_y, A_x, respectively.

Then, if a perpendicular (ρ) be dropped from the origin upon the given area, we have,

$$A_x = A \cos \rho, x, \quad A_y = A \cos \rho, y, \quad A_z = A \cos \rho, z.$$

since by the principles of Geometry, the projection of an area upon any plane is equal to the area multiplied by the cosine of its inclination to that plane.

Therefore, taking the sum of the squares of these projections,

$$A^2_x + A^2_y + A^2_z = A^2 \{\cos^2 \rho, x + \cos^2 \rho, y + \cos^2 \rho, z\}$$
$$= A^2 \quad (244).$$

278. Cor. 1. Hence, in a triangular pyramid, which has three contiguous edges at right angles to one another, the square of the face subtending the trihedral angle, formed by the planes of these edges, is equal to the sum of the squares of the three faces which contain the same angle.

For the three last faces are the projections of the first upon the planes formed by the rectangular edges.

This property of the pyramid is analogous to the celebrated property of the right-angled triangle in Plane Geometry.

279. Cor. 2. Let $x'y'$, $x'z'$, $y'z'$ be any other three rectangular planes, originating at the same point as before, then, if the projections of A upon these planes be denoted by $A_{x'}$ $A_{y'}$ $A_{z'}$ respectively, it may be proved precisely as above, that

$$A^2 = A^2_{x'} + A^2_{y'} + A^2_{z'};$$
$$\therefore\ A^2_x + A^2_y + A^2_z = A^2_{x'} + A^2_{y'} + A^2_{z'}.$$

Hence, it follows that *the sum of the squares of the projections of an area upon any three co-ordinate rectangular planes is constant.*

Note. When we speak of the *square of an area*, the expression is to be understood in a numerical sense; a geometrical signification, however, may be annexed to it, by substituting for the areas in question straight lines that are proportional to them.

280. Prop. 2. The same notation being used, it is required to prove that the projection of A upon any plane is

equal to the sum of the projections of A_x A_y A_z upon the same plane.

Let the plane of $z'y'$ be that on which A is projected.

Then $A_{x'} = A \cos \rho, x$.

Now

$$\cos \rho, x' = \cos \rho, x \cos x', x + \cos \rho, y \cos x', y + \cos \rho, z \cos x', z;$$

therefore, multiplying each term by A,

$$A \cos \rho, x' = A \cos \rho, x \cos x', x + A \cos \rho, y \cos x', y + A \cos \rho, z \cos x', z;$$

hence, substituting for $A \cos \rho, x$, $A \cos \rho, y \ldots$ their values in Prop. 2, we have

$$A_{x'} = A_x . \cos x', x + A_y \cos x', y + A_z \cos x', z,$$

as was to be proved.

281. Cor. 1. Let $A, A', A'' \ldots$ be any number of areas in different planes,

$X, Y, Z \ldots$ the sum of their projections on zy, xz, xy,

$X', Y', Z' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ldots z'y', x'z', x'y'$.

Then, it follows directly from what has just been proved, that

$$X' = X \cos x', x + Y \cos x', y + Z \cos x', z,$$

$$Y' = X \cos y', x + Y \cos y', y + Z \cos y', z,$$

$$Z' = X \cos z', x + Y \cos z', y + Z \cos z', z.$$

282. Cor. 2. Since X, Y, Z, and X', Y', Z' are the projections on two systems of co-ordinate rectangular planes, we have, conversely,

$$X = X' \cos x, x' + Y' \cos x, y' + Z' \cos x, z',$$

$$Y = X' \cos y, x' + Y' \cos y, y' + Z' \cos y, z',$$

$$Z = X' \cos z, x' + Y' \cos z, y' + Z' \cos z, z'.$$

283. **Prop. 3.** It is required to prove that

$$X^2 + Y^2 + Z^2 = X'^2 + Y'^2 + Z'^2.$$

Squaring each line in Cor. 1. Prop. 2. and adding the results *vertically*, we have

$$\begin{aligned} X'^2 + Y'^2 + Z'^2 = {} & X^2 \{\cos^2 x', x + \cos^2 x', y + \cos^2 x', z\} \\ & + Y^2 \{\cos^2 y', x + \cos^2 y', y + \cos^2 y', z\} \\ & + Z^2 \{\cos^2 z', x + \cos^2 z', y + \cos^2 z', z\} \\ + 2XY & \{\cos x', x \cos x', y + \cos y', x \cos y', y + \cos z', x \cos z', y\} \\ + 2XZ & \{\cos x', x \cos x', z + \cos y', x \cos y', z + \cos z', x \cos z', z\} \\ + 2ZY & \{\cos x', z \cos x', y + \cos y', z \cos y', y + \cos z', y \cos z', z\}. \end{aligned}$$

Now, the coefficients of X^2, Y^2, Z^2, in the three first lines, are (244) each $= 1$, and the coefficients of XY, XZ, ZY, in the last three lines, are (247) each $= 0$;

$$\therefore X'^2 + Y'^2 + Z'^2 = X^2 + Y^2 + Z^2.$$

This sum, being independent of the three planes of projection, is invariable.

284. **Cor. 1.** Hence

$$X' = \sqrt{\{X^2 + Y^2 + Z^2 - (Y'^2 + Z'^2)\}},$$

which quantity therefore is a *maximum* when Y' and Z' each $= 0$; therefore, in this case,

$$X' = \sqrt{\{X^2 + Y^2 + Z^2\}}.$$

The plane $z'y'$ is generally called *the plane of greatest projection.*

285. **Cor. 2.** The projections of each of the areas A, A'... and consequently, the sums of those projections being the same on all planes that are parallel to one another, it follows that the absolute position in space of the plane of greatest projection is indeterminate. Its position, however, relatively to the three primitive planes, may easily be found, by means of Cor. 2. Prop. 2.

For, since Y' and Z' each $= 0$,

$$\cos x', x = \frac{X}{X'} = \therefore \frac{X}{\sqrt{\{X^2+Y^2+Z^2\}}}.$$

$$\text{Similarly, } \cos x', y = \frac{Y}{\sqrt{\{X^2+Y^2+Z^2\}}},$$

$$\text{and } \cos x', z = \frac{Z}{\sqrt{\{X^2+Y^2+Z^2\}}}.$$

286. **Prop. 4.** The sum of the projections of the areas A, A', A''... is the same on all planes that are equally inclined to the plane of greatest projection.

Let ρ be the perpendicular dropped from the origin upon any plane.

Then, the angle ρ, x' measures its inclination to the plane of greatest projection.

Let V be the sum of the projections on this plane, then

$$V = X \cos \rho, x + Y \cos \rho, y + Z \cos \rho, z,$$

$$\text{but } X = X' \cos x', x, \quad Y = X' \cos x', y, \quad Z = X' \cos x', z;$$

therefore, by substitution,

$$V = X' \{\cos \rho, x \cos x', x + \cos \rho, y \cos x', y + \cos \rho, z \cos x', z\}$$
$$= X' . \cos \rho, x'.$$

Hence, since X' is constant (284.), the value of V depends solely upon $\cos \rho, x'$, and therefore is the same for all planes that form the same angle with the plane of greatest projection: which was to be proved.

Observation. The properties of projection, which have been proved true of any plane figures, are true also of straight lines projected on three rectangular axes.

CHAP. VIII.

ON THE EQUATION TO THE PLANE WHICH PASSES THROUGH THREE GIVEN POINTS.

287. **Prop. 1. To find the equation to the plane which passes through three given points.**

Let x', y', z', be the co-ordinates of the first point,

x'', y'', z'', those of the second,

x''', y''', z''', those of the third.

The equation sought will be of the form

$$Ax+By+Cz=D \ldots\ldots\ldots (1).$$

The substitution of the given co-ordinates for x, y, z in (1) furnishes the three following equations:

$$Ax'+By'+Cz'=D,$$

$$Ax''+By''+Cz''=D.$$

$$Ax'''+By'''+Cz'''=D.$$

Whence, deducing by the ordinary method of elimination, the values of $\frac{A}{D}$, $\frac{B}{D}$, $\frac{C}{D}$, and assuming the common denominator $=D$, we have

$$A = z'y''' - z'''y' + z''y''' - z'''y'' + z''y' - z'y'',$$

$$B = x'z''' - x'''z' + x''z''' + x'''z'' + x''z' - x'z'',$$

$$C = y'x''' - y'''x' + y''x''' - y'''x'' + y''x' - y'x'',$$

$$D = x'y''z''' - x'y'''z'' + x''y'''z' + x''y'z''' + x'''y'z'' - x'''y''z'.$$

Now A is equal to twice the area of a triangle, in the plane of yz, the co-ordinates of whose angular points are

$$y', z'; \; y'', z''; \; y''', z''',$$

Art. 64. Cor.; and this is evidently the projection on yz of the triangle in space, which is formed by joining the three given points.

Hence, if a denote the triangle in space, a_x, a_y, a_z its respective projections, and ρ the perpendicular dropped upon it from the origin, we have

$$\text{and similarly,} \quad \left.\begin{aligned} A &= 2a_x \\ B &= 2a_y \\ C &= 2a_z \end{aligned}\right\} \ldots\ldots\ldots\ldots (2).$$

$$\text{Again, } \rho = \frac{D}{\sqrt{(A^2 + B^2 + C^2)}} \quad (265),$$

$$\text{but } a^2 = a_x^2 + a_y^2 + a_z^2 \quad (277),$$

$$= \tfrac{1}{4}(A^2 + B^2 + C^2) \text{ from } (2);$$

$$\therefore \sqrt{(A^2 + B^2 + C^2)} = 2a;$$

therefore, by substitution, $\rho = \dfrac{D}{2a}$, or $D = 2a\rho \ldots\ldots (3)$,

whence, replacing the indeterminate coefficients A, B, C, D

by their values in (2) and (3), the equation to the plane passing three given points finally becomes

$$2a_x.x+2a_y.y+2a_z.z=2a\rho,$$

$$\text{or } a_x.x+\ a_y.y+\ a_z.z=\ a\rho\ldots\ldots(4).$$

288. Cor. 1. The volume of a triangular pyramid being equal to $\frac{1}{3}$ of the area of the base multiplied by the altitude, it follows from (3), that $D=$ six times the volume of the pyramid, whose vertex is the origin, and whose base is the triangle in space.

289. Cor. 2. From equation (4), when translated into geometrical language, is deduced the following theorem:

If any point be assumed in the plane which passes through a given triangle situated in space and referred to three rectangular planes, the volume of the pyramid whose vertex is the origin, and whose base is the given triangle, is equal to the sum of the volumes of the three pyramids, whose vertices are the assumed point, and whose bases are the projections of the given triangle on the planes of xy, xz and yz.

For, let P, $(x, y, z,)$ be the assumed point; then, retaining the same notation as in the proposition, $a_x.x=$ three times the volume of the pyramid, whose base is the projection on yz, and whose altitude is x, or vertex P.

The value of the remaining terms of (4), may be similarly expressed. Whence the truth of the proposition.

For this elegant theorem we are indebted to Monge, who gave a demonstration of it, on other principles, in the *Journ. de l'Ecole Polytech.* 14^e. Cah.

290. It may not be superfluous to present another proof of Monge's theorem, which has the advantage of being independent of the foregoing proposition.

The same notation being retained,

$$x \cos \rho, x + y \cos \rho, y + z \cos \rho, z = \rho,$$

is the equation to any plane.

Multiplying each member of this equation by a, we have

$$ax \cos \rho, x + ay \cos \rho, y + az \cos \rho, z = a\rho;$$

but (277),

$$a \cos \rho, x = a_x, \quad a \cos \rho, y = a_y, \quad a \cos \rho, z = a_z,$$

therefore, by substitution,

$$a_x . x + a_y . y + a_z . z = a\rho,$$

as was to be proved.

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CHAP. IX.

ON THE STRAIGHT LINE WHEN REFERRED TO OBLIQUE CO-ORDINATES.

291. With the view of avoiding unnecessary detail, we have hitherto confined our attention to the case in which the co-ordinate planes are rectangular. But although it is seldom requisite to use oblique co-ordinates, yet cases sometimes occur, in which the employment of them leads to elegant and useful results. We propose, therefore, to allot this and the next Chapter to the consideration of the straight line and plane, when they are referred to any three co-ordinate planes whatever.

292. If ρ be any straight line in space, and ρ_y, ρ_x denote its projections on the planes of xz and yz respectively, then the equations to ρ will retain the same form as before,

$$x = az + \alpha,$$

$$y = bz + \beta,$$

in which α, β have their former acceptation, (231), and

$$a = \frac{\sin \rho_y, z}{\sin \rho_y, xy}, \quad b = \frac{\sin \rho_x, z}{\sin \rho_x, xy}.$$

293. Let $AP = \rho$ be any straight line passing through the origin, AM, MN, NP its co-ordinates, (fig. 79.): from N, P drop the perpendiculars Nn, Pp on the plane of yz, join np, and draw NQ parallel to AX.

$$\text{Then } Nn = NQ \sin NQN = x \sin x, yz,$$

$$Pp = AP \sin PAp = \rho \sin \rho, yz,$$

but $Nn = Pp$, since Np is evidently a rectangle;

$$\therefore\ x \sin x, yz = \rho \sin \rho, yz,$$

$$\text{or } x = \frac{\sin \rho, yz}{\sin x, yz}\rho \ldots\ldots (1).$$

In like manner it may be proved, that

$$y = \frac{\sin \rho, xz}{\sin y, xz}\rho \ldots\ldots (2),$$

$$\text{and } z = \frac{\sin \rho, xy}{\sin z, xy}\rho \ldots\ldots (3).$$

Hence, if the coefficients of ρ in (1), (2) and (3), respectively, be denoted by m, n and p, there will result

$$x = m\rho,\ y = n\rho,\ z = p\rho.$$

These are the equations which are commonly used to characterize a straight line in space, when referred to oblique co-ordinates.

294. From Art. (292.) it appears, that the form of the equations to a straight line is not affected by the inclination of the axes; whence it is evident, that the equations to a line passing through *one* or through *two* given points, and the conditions of parallelism or of intersection of two lines, will occur under the same form, whether the co-ordinates be oblique or rectangular. It will be necessary, therefore, to direct our inquiries to those propositions alone, which involve the consideration of *distance* or of *inclination*.

295. PROP. 1. To find the distance of any point from the origin.

Let P be the point, AX, AY, AZ the axes to which it is referred; (fig. 80.). From M, N drop the perpendiculars Mm, Nn upon AZ, and draw Mp parallel to AZ.

$$\text{Let } AP = \rho,\ AM = x,\ MN = y,\ NP = z,$$
$$\text{Then } AP^2 = AN^2 + NP^2 + 2\,AN\,.\,NP \cos ZAN.$$

$$\text{But } AN^2 = AM^2 + MN^2 + 2AM.MN \cos YAX$$

$$= x^2 + y^2 + 2xy \cos x, y.$$

And $AN \cos ZAN = An = Am + Mq = x \cos x, z + y \cos y, z$;

therefore by substitution in (1), we have

$$\rho^2 = x^2 + y^2 + z^2 + 2xy \cos x, y + 2xz \cos x, z + 2yz \cos y, z.$$

Whence the distance required is known.

296. Cor. 1. Let $AMPC$ be an oblique parallelepiped, then by the proposition just proved, (fig. 81.),

$$AP^2 = x^2 + y^2 + z^2 + 2xy \cos x, y + 2xz \cos x, z + 2yz \cos y, z;$$

also

$$MC^2 = x^2 + y^2 + z^2 - 2xy \cos x, y - 2xz \cos x, z + 2yz \cos y, z,$$

$$BQ^2 = x^2 + y^2 + z^2 - 2xy \cos x, y + 2xz \cos x, z - 2yz \cos y, z,$$

$$ND^2 = x^2 + y^2 + z^2 + 2xy \cos x, y - 2xz \cos x, z - 2yz \cos y, z;$$

$$\therefore AP^2 + MC^2 + BQ^2 + ND^2 = 4(x^2 + y^2 + z^2).$$

Wherefore, *In an oblique parallelepiped, the sum of the squares of the four diagonals, is equal to the sum of the squares of the twelve edges.*

297. Cor. 2. Since $x = m\rho$, $y = n\rho$, $z = p\rho$, we have, on substituting, and dividing each side by ρ^2,

$$m^2 + n^2 + p^2 + 2mn \cos x, y + 2mp \cos x, z + 2np \cos y, z = 1 \textit{ to rad. } 1 \ldots (1),$$

$$= r^2 \textit{ to rad. } r \ldots (2).$$

The same relation subsists among the coefficients m', n', p' of any other line ρ'.

298. Prop. 2. To find the distance between any two points in space.

Let P and P' be any two points, x', y', z' the co-ordinates of the latter, and suppose $PP' = \delta$.

Then δ may be considered as the diagonal of a parallelepiped, whose edges are

$$x'-x,\ y'-y,\ z'-z;$$

therefore (295),

$$\delta^2=(x'-x)^2+(y'-y)^2+(z'-z)^2+2\,(x'-x)\,(y'-y)\cos x,y$$
$$+2\,(x'-x)\,(z'-z)\cos x,z+2\,(y'-y)\,(z'-z)\cos y,z.$$

Whence the distance sought is known.

299. Cor. Expanding the right hand member of the equation, reducing it by means of (295), and replacing δ^2 by its value $\rho^2+\rho'^2-2\rho\rho'\cos\rho,\rho'$, we have

$$\begin{aligned}\rho^2+\rho'^2-2\rho\rho'\cos\rho,\rho'=\rho^2+\rho'^2&-2\,\{xx'+(yz'+zy')\cos y,z\}\\&-2\,\{yy'+(xz'+zx')\cos x,z\}\\&-2\,\{zz'+(xy'+yx')\cos x,y\}.\end{aligned}$$

300. Prop. 3. To find the inclination of two straight lines in space.

Through the origin draw two lines AP, AP' parallel respectively to the given lines, and denote them by ρ,ρ'; then the angle ρ,ρ' will be the inclination sought.

For x, x' . . . in the last Cor., let $m\rho$, $m'\rho'$. . . be substituted; and let the angles x, y; x, z; y, z be denoted by a, β, γ, respectively; the result thus obtained, when divided by $2\rho\rho'$, becomes

$$\cos\rho'\rho=$$

$${}'+nn'+pp'+(mn'+nm')\cos a+(mp'+pm')\cos\beta+(np'+pn')\cos\gamma \text{ to rad. } 1\,.\,(1$$

$$\text{or}=$$

$$\frac{mm'+nn'+pp'+(nm'+mn')\cos a+(mp'+pm')\cos\beta+(np'+pn')\cos\gamma}{\sqrt{\{m^2+n^2+p^2+2mn\cos a+\,.\,.\}}\ \sqrt{\{m'^2+n'^2+p'^2+2m'n'\cos a+\,.\,.\}}}$$

$$\text{to rad. } r\,.\,.\,.\,.\,.\,.\,.\,.\,(2).$$

301. Cor. 1. When the axes are rectangular,

$$\cos x,y,\ \cos x,z,\ \text{and}\ \cos y,z\ \text{each} = 0,$$

and we again fall upon the expression for $\cos\rho,\rho'$ in Art. (248).

302. Cor. 2. The former of these values of $\cos\rho,\rho'$ may be written as follows:

$$\begin{aligned}\cos\rho,\rho' = {} & m'\{m+n\cos x,y+p\cos x,z\}\\ & +n'\{n+m\cos y,x+p\cos y,z\}\\ & +p'\{p+m\cos z,x+n\cos z,y\}.\end{aligned}$$

Now, if from P perpendiculars be dropped upon each of the axes, Pp being that upon AX, then, (fig. 82.)

$$Ap = AM + Mm + Nn,$$

$$\begin{aligned}\text{or}\quad \rho\cos\rho,x & = x+y\cos x,y+z\cos x,z\\ & = m\rho+n\rho\cos x,y+p\rho\cos x,z;\end{aligned}$$

$$\therefore \cos\rho,x = m+n\cos x,y+p\cos x,z.$$

$$\begin{aligned}\text{Similarly,}\quad \cos\rho,y & = n+m\cos x,y+p\cos y,z,\\ \cos\rho,z & = p+m\cos x,z+n\cos z,y;\end{aligned}$$

$$\left.\begin{aligned}\therefore \cos\rho,\rho' & = m'\cos\rho,x+n'\cos\rho,y+p'\cos\rho,z\\ \text{or } & = m\cos\rho',x+n\cos\rho',y+p\cos\rho'z\end{aligned}\right\}\ldots(1).$$

303. Cor. 3. If for m, m'.. their values (293) be substituted, then

$$\cos\rho,\rho' = \frac{\sin\rho',yz}{\sin x,yz}\cos\rho,x + \frac{\sin\rho',xz}{\sin y,xz}\cos\rho,y + \frac{\sin\rho',xy}{\sin z,xy}\cos\rho,z,$$

$$\text{or } = \frac{\sin\rho,yz}{\sin x,yz}\cos\rho',x + \frac{\sin\rho,xz}{\sin y,xz}\cos\rho',y + \frac{\sin\rho,xy}{\sin z,xy}\cos\rho',z.$$

CHAP. X.

ON THE PLANE WHEN REFERRED TO OBLIQUE AXES.

304. Whatever be the system of co-ordinates employed, the equation to the plane presents itself always under the same form

$$\rho = x \cos \rho, x + y \cos \rho, y + z \cos \rho, z,$$

or, if $M = \cos \rho, x$, $N = \cos \rho, y$, $P = \cos \rho, z$,

$$\rho = Mx + Ny + Pz.$$

We shall, therefore, confine our attention, as in the case of the straight line, to those propositions alone which are affected by a change in the inclination of the co-ordinate planes.

305. Prop. 1. Given the equation to any plane Π, to find its inclination to each of the co-ordinate planes.

Let the given equation be

$$p = Mx + Ny + Pz.$$

Now, in general (303),

$$\cos \rho, \rho' = \frac{\sin \rho', yz}{\sin x, yz} \cos \rho, x + \frac{\sin \rho', xz}{\sin y, xz} \cos \rho, y + \frac{\sin \rho', xy}{\sin z, xy} \cos \rho, z.$$

Suppose that ρ' is perpendicular successively to the planes of yz, xz, xy; then, substituting for $\cos \rho, x$; $\cos \rho, y$, and $\cos \rho, z$, their values M, N, and P, we have

$$\cos \Pi, yz = \frac{1}{\sin x, yz} M + \frac{\cos yz, xz}{\sin y, xz} N + \frac{\cos yz, xy}{\sin z, xy} P,$$

$$\cos \Pi, xz = \frac{1}{\sin y, xz} N + \frac{\cos xz, yz}{\sin x, yz} M + \frac{\cos xz, xy}{\sin z, xy} P,$$

$$\cos \Pi, xy = \frac{1}{\sin z, xy} P + \frac{\cos xy, yz}{\sin x, yz} M + \frac{\cos xy, xz}{\sin y, xz} N.$$

The inclinations sought are therefore known.

306. Cor. Let the co-ordinate planes be rectangular.

Then $\sin x, yz = 1$, $\cos xz, yz = 0$,

and the same being true with respect to the other trigonometrical functions involved in each equation, we have

$$\cos \Pi, yz = M$$

$$= \therefore \cos \rho, x,$$

which it ought to be; since ρ being perpendicular to Π, and x to the plane of yz, the angle ρ, x measures the inclination of the two planes.

307. Prop. 2. To find the relation that subsists among M, N, P, the coefficients in the equation to a plane.

Since $\rho = Mx + Ny + Pz = \{Mm + Nn + Pp\} \rho$,

we have, on dividing by ρ,

$$Mm + Nn + Pp = 1 \ldots\ldots (1);$$

or, substituting for m, n, p, their values (293.)

$$\frac{\sin \rho, yz}{\sin x, yz} M + \frac{\sin \rho, xz}{\sin y, xz} N + \frac{\sin \rho, xy}{\sin z, xy} P = 1,$$

but $\sin \rho, yz = \cos \Pi, yz$, and so on;

$$\therefore \frac{\cos \Pi, yz}{\sin x, yz} M + \frac{\cos \Pi, xz}{\sin y, xz} N + \frac{\cos \Pi, xy}{\sin z, xy} P = 1.$$

Whence, replacing $\cos \Pi, yz$, $\cos \Pi, xz$, and $\cos \Pi, xy$, by their values given above (305.), we have, on reducing the result,

$$\frac{M^2}{\sin^2 x, yz} + \frac{N^2}{\sin^2 y, xz} + \frac{P^2}{\sin^2 z, xy} + \frac{2MN}{\sin x, yz \sin y, xz} \cos yz, xz + \frac{2MP}{\sin x, yz \sin z, xy} \cos yz, xy + \frac{2NP}{\sin y, xz \sin z, xy} \cos xz, xy,$$

$$= 1 \textit{ to radius } 1 \ldots\ldots\ldots (1),$$

$$\text{or } = r^2 \textit{ to radius } r \ldots\ldots\ldots (2),$$

either of which equations expresses the relation sought.

308. Cor. Let the co-ordinate planes be rectangular;

$$\text{then } M^2 + N^2 + P^2 = 1 \textit{ to radius } 1,$$

$$\text{or } = r^2 \textit{ to radius } r,$$

as in Art. 271.

309. Prop. 3. To find the angle of inclination of two planes Π and Π'.

If planes parallel to the given planes be drawn through the origin, their equations will be

$$Mx + Ny + Pz = \rho,$$

$$M'x + N'y + P'z = \rho'$$

and, since $\angle\, \Pi, \Pi' = \angle\, \rho, \rho'$, we have, (303)

$$\cos \Pi, \Pi' = \frac{\sin \rho, yz}{\sin x, yz} \cos \rho', x + \frac{\sin \rho, xz}{\sin y, xz} \cos \rho', y + \frac{\sin \rho, xy}{\sin z, xy} \cos \rho', z = \frac{\cos \Pi, yz}{\sin x, yz} M' + \frac{\cos \Pi, xz}{\sin y, xz} N' + \frac{\cos \Pi, xy}{\sin z, xy} P'.$$

Whence, substituting for $\cos \Pi, yz$, $\cos \Pi, xz$, and $\cos \Pi, xy$, their values in (305), we have, on completing the operation,

$$\cos \Pi, \Pi' =$$

$$\frac{MM'}{\sin^2 x,yz}+\frac{NN'}{\sin^2 y,xz}+\frac{PP'}{\sin^2 z,xy}+\left(\frac{MN'}{\sin x,yz}+\frac{NM'}{\sin y,xz}\right)\cos yz,xz+\left(\frac{MP'}{\sin x,yz}+\frac{PM'}{\sin z,xy}\right)\cos yz,xy+\left(\frac{NP'}{\sin y,xz}+\frac{PN'}{\sin z,xy}\right)\cos xz,xy,$$

to radius 1 . . . . (1).

or =

$$\pm\frac{\dfrac{MM'}{\sin^2 x,yz}+\dfrac{NN'}{\sin^2 y,xz}+\dfrac{PP'}{\sin^2 z,xy}+\left(\dfrac{MN'}{\sin x,yz}+\dfrac{NM'}{\sin y,xz}\right)\cos yz,xz+\left(\dfrac{MP'}{\sin x,yz}+\dfrac{PM'}{\sin z,xy}\right)\cos yz,xy+\left(\dfrac{NP'}{\sin y,xz}+\dfrac{PN'}{\sin z,xy}\right)\cos xz,xy}{\sqrt{\left\{\dfrac{M^2}{\sin^2 x,yz}+\dfrac{N^2}{\sin^2 y,xz}+\dfrac{P^2}{\sin^2 z,xy}+\dfrac{2MN}{\sin x,yz \sin y,xz}\cos yz,xz+\dfrac{2MP}{\sin x,yz \sin z,xy}\cos yz,xy+\dfrac{2NP}{\sin y,xz \sin z,xy}\cos xz,xy\right.}}$$

to radius r . . . . (2).

For a very elegant investigation of this formula, the reader is referred to a paper by Mr. Whewell in the second volume of the Cambridge Philosophical Transactions.

310. Cor. 1. When the planes are at right angles to each other, the numerator of the expression just deduced $=0$.

311. **Cor. 2.** Let the co-ordinate planes be rectangular;

$$\text{then } \cos \Pi, \Pi' = MM' + NN' + PP' \text{ . . . . . } \textit{to rad. } 1,$$

$$\text{or } = \pm \frac{MM' + NN' + PP'}{\sqrt{(M^2+N^2+P^2)(M'^2+N'^2+P'^2)}} \textit{ to rad. } r, \text{ as in Art. 274.}$$

312. **Prop. 4.** To find the inclination of a straight line to a plane.

We shall denote the proposed line by ϱ, in order to distinguish it from the line ρ, which is appropriated to signify the perpendicular to the plane Π.

Now, in general,

$$\cos \rho, \rho' = mM' + nN' + pP' \quad (303)$$

$$\therefore \sin \rho, \Pi' = mM' + nN' + pP'.$$

Hence, ϱ being any line, and Π any plane to which it is inclined, we have

$$\sin \varrho, \Pi = mM + nN + pP \textit{ to radius } 1 \text{ . . . . } (1),$$

$$\text{or } = \frac{mM + nN + pP}{\sqrt{\{m^2+n^2+p^2+2mn \cos x, y + \ldots\}} \sqrt{\left\{\frac{M^2}{\sin^2 x, yz} + \frac{N^2}{\sin^2 y, xz} + \frac{P^2}{\sin^2 z, xy} + \frac{2MN}{\sin x, yz \sin y, xz} \cos yz, xz \ldots\right\}}},$$

$$\textit{to radius } r \text{ . . . . } (2).$$

313. Cor. 1. When the line is perpendicular to the plane,

$$mM + nN + pP = 1.$$

314. Cor. 2. Let the axes be rectangular;

then, $\sin \rho, \Pi = \dfrac{mM + nN + pP}{\sqrt{(m^2 + n^2 + p^2)}\,\sqrt{(M^2 + N^2 + P^2)}}.$

CHAP. XI.

ON THE TRANSFORMATION OF CO-ORDINATES IN SPACE.

315. **Prop. 1.** To pass from one system of oblique axes to another, the origin being the same.

Let x, y, z and x', y', z' be the co-ordinates of a point referred to the primitive and new axes respectively.

If the same reasoning be applied to the plane, which was used (72) in the case of the straight line, it may be proved that the primitive co-ordinates are *linear* functions of the new ones; we shall assume, therefore,

$$x = mx' + ny' + pz',$$

$$y = m'x' + n'y' + p'z',$$

$$z = m''x' + n''y' + p''z',$$

in which the constants m, n, p ...... are now to be determined.

Suppose that y' and z' each $= 0$, or that the point is on AX',

$$\text{then } m = \frac{x}{x'} = \frac{\sin x', yz}{\sin x, yz},$$

by Art. (293), since ρ is here $= x'$;

$$m' = \frac{y}{x'} = \frac{\sin x', xz}{\sin y, xz},$$

$$m'' = \frac{z}{x'} = \frac{\sin x', xy}{\sin z, xy}.$$

In like manner, by supposing the point to be successively on the axes AY', AZ', we may obtain the values of n, n', n'', and of p, p', p''. We have, therefore, by substitution,

$$\left.\begin{aligned} x &= \frac{1}{\sin x, yz}\{x' \sin x', yz + y' \sin y', yz + z' \sin z', yz\} \\ y &= \frac{1}{\sin y, xz}\{x' \sin x', xz + y' \sin y', xz + z' \sin z', xz\} \\ z &= \frac{1}{\sin z, xy}\{x' \sin x', xy + y' \sin y', xy + z' \sin z', xy\} \end{aligned}\right\} \ldots (1).$$

From the symmetrical form of these expressions, they may easily be remembered.

316. Cor. 1. Let the primitive axes be rectangular, and the new ones oblique.

Then the denominator in each of the foregoing formulas becomes $= 1$; also

$$\sin x', yz = \cos x', x,$$

and similarly for the remaining terms; we therefore have

$$\left.\begin{aligned} x &= x' \cos x', x + y' \cos y', x + z' \cos z', x \\ y &= x' \cos x', y + y' \cos y', y + z' \cos z', y \\ z &= x' \cos x', z + y' \cos y', z + z' \cos z', z \end{aligned}\right\} \ldots\ldots (2),$$

and since the primitive system is rectangular, we have the following equations:

$$\left.\begin{aligned} \cos^2 x', x + \cos^2 x', y + \cos^2 x', z &= 0 \\ \cos^2 y', x + \cos^2 y', y + \cos^2 y', z &= 0 \\ \cos^2 z', x + \cos^2 z', y + \cos^2 z', z &= 0 \end{aligned}\right\} \ldots\ldots (a).$$

Hence, of the *nine* angles involved in (3), six only are independent, since equation (a) gives three of them.

317. Cor. 2. Let both systems be rectangular.

Then, since x' and y', x' and z', y' and z', are at right angles to each other, we have, (247),

$$\left.\begin{array}{l}\cos x',x\cos y',x+\cos x',y\cos y',y+\cos x',z\cos y',z=0\\ \cos x',x\cos z',x+\cos x',y\cos z',y+\cos x',z\cos z',z=0\\ \cos y',x\cos z',x+\cos y',y\cos z',y+\cos y',z\cos z',z=0\end{array}\right\}\ldots(\beta).$$

These equations (β), together with the other equations of condition (α), reduce to *three* the nine constant coefficients which enter into the general formulas (2); whence it appears, that three independent angles are sufficient for executing the proposed transformation. That it requires, indeed, only three angles to determine the position of the new axes may readily be shewn.

Let AX, AY, AZ and AX', AY', AZ' be the two proposed systems; (t) the trace of xy on $x'y'$, and the $\angle\, x'y', xy$ the inclination of these two planes.

Then the position of the second system relatively to that of the first, will depend on the three angles

$$x'y',xy;\quad t,x \text{ and } t,x'.$$

For, the new axes being rectangular, if the position of any two of them, AX', AY' for example, be given, that of the third AZ' will necessarily be known; also, since AX', AY' are at right angles to each other, the direction of the latter is deducible from that of the former, when the plane $x'y'$, in which both are situated, is given. Hence, the relative position of the new axes, with respect to the primitive ones, will be determined when the position of $x'y'$ and of Ax' is known; that is, when the angles

$$x'y',\ xy;\quad t,x \text{ and } t,x', \text{ are known.}$$

318. Prop. 2. To pass from a given system of rectangular axes to another, whose position, relatively to that of the former, is known.

The three angles necessary for fixing the position of the new axes are (317),

$$x'y', x, y; \qquad t, x; \qquad \text{and } t, x'.$$

The aim, therefore, of the problem, is to express in terms of these angles, the nine angles of (2), namely,

$$\begin{array}{lll} x', x; & x', y; & x', z, \\ y', x; & y', y; & y', z, \\ z', x; & z', y; & z', z. \end{array}$$

I. To express the first system x', x; x', y and x', z.

Let AX' be the axis of x', and AT the trace (t) of $x'y'$ upon xy, (fig. 83).

In AX' take $Ax' = 1$, and conceive it to be the radius of a sphere, whose centre is A, and which cuts the planes of

$$x'x, \quad x'y, \quad x'z, \quad \text{and of } tx, \quad tx',$$

in the arcs of great circles,

$$x'x, \quad x'y, \quad x'z, \quad \text{and} \quad tx, \quad tx'.$$

Then these arcs will respectively measure the angles

$$x', x; \quad x', y; \quad x', z, \quad \text{and } t, x; \quad t, x'.$$

Also the $\angle\, x'tx$ will equal the inclination of the planes $x'y'$, xy. Let $Zx'z$ be an arc of 90° passing through x', and meeting xy in z.

Then, since in any spherical triangle

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

we have

(1) In the triangle $x'xt$,

$$\cos x', x = \cos t, x \cos t, x' + \sin t, x \sin t, x' \cos x'y', xy \dots (1).$$

(2) In the triangle $x'yt$,

$$\text{(observing that } \angle\, x'ty = \pi - x'y', xy,$$

$$\text{and } \angle\, t, y = \frac{\pi}{2} - t, x\text{)},$$

$$\cos x', y = \sin t, x \cos t, x' - \sin t, x \cos t, x' \cos x'y', xy \;\ldots\; (2).$$

(3) In the triangle $x'zt$, *right-angled at* z, by Napier's rules,

$$\cos x', z = \sin t, x' \,.\, \sin x'y', xy \;\ldots\ldots\ldots\ldots\; (3).$$

II. To express the second system y', x; y', y and y', z.

Let AY' be the axis of y', (fig. 84).

Then, constructing and reasoning precisely as before, we prove, without difficulty, that

(1) In the triangle $y'xt$,

$$\text{(observing that } \angle\, t, y' = \frac{\pi}{2} + t, x'\text{)},$$

$$\cos y', x = -\cos t, x \sin t, x' + \sin t, x \cos t, x' \cos x'y, xy \;\ldots\; (4).$$

(2) In the triangle $y'yt$,

$$\cos y', y = -\sin t, x \cos t, x' - \cos t, x \cos t, x' \cos x'y', xy \;\ldots\; (5).$$

(3) In the triangle $y'xt$ *right-angled at* z,

$$\cos y', z = \sin t, x \cos t, x \;\ldots\ldots\ldots\ldots\ldots\ldots\; (6).$$

III. To express the third system $z'x$; z', y and z', z.

Let AZ' be the axis of z', (fig. 85).

Then, as in the first two cases, we have

(1) In the triangle $z'xt$, *right-angled at* z',

$$\cos z', x = -\sin t, x \sin t, x' \;\ldots\ldots\; (7).$$

(2) In the triangle $z'yt$, *right-angled at* z',

$$\cos z',y=\sin t,x\ \cos t,x' \ldots\ldots\ldots (8).$$

(3) The axes of z and of z' being perpendicular to the planes of xy and of $x'y'$, we have

$$\angle z',z=\angle x'y',xy\text{; and, therefore,}$$

$$\cos z',z=\cos x'y',xy \ldots\ldots\ldots\ldots (9).$$

Substituting these nine values formulas (2) of Art. 316, and reducing the results, we fina tain

$$x=x'\{\cos t,x\cos t, \quad t,x\sin t,x'\cos x'y',xy\}$$
$$+y'\{-\cos t,x\sin t, \quad t,x\cos t,x'\cos x'y',xy\}$$
$$-z'\sin t,x\sin x'y',$$
$$y=x'\{\sin t,x\cos t,x'- \quad \sin t,x'\cos x'y',xy\}$$
$$-y'\{\sin t,x\sin t,x'+ \quad \cos t,x'\cos x'y',xy\}$$
$$+z'\cos t,x\sin x'y',xy$$
$$z=x'\sin t,x'\sin x'y',xy+y'\cos t,x'\sin x'y',xy$$
$$+z'\cos x'y',xy,$$

which are the formulas required.

319. COR. These formulas may be considerably simplified, if the trace AT be assumed as the new axis of x'. In this case AX' coinciding with AT, the $\angle t,x'=0$; therefore, making $\sin t,x'=0$, and $\cos t,x'=1$, the reduced formulas are

$$x=x'\cos t,x+y'\sin t,x\cos x'y',xy,$$

$$y=x'\sin t,x-y'\cos t,x\cos x'y',xy+z'\cos t,x\sin x'y',xy,$$

$$z=y'\sin x'y',xy+z'\cos x'y',xy.$$

Observation. The transformation of co-ordinates from one rectangular system to another, is subservient to various important inquiries in the higher departments of Mechanics. Among the different methods which have been proposed for executing this operation, the one above given is the most generally used. It was employed for the first time by Euler, and was afterwards adopted by Lagrange and Laplace.

320. If the *origin*, as well as direction, of the axes be changed, let a, b, c be the co-ordinates of the new origin; then it is only necessary to add a, b, c respectively to the values of x, y, z in the general formulas (2).

When the origin alone is displaced, and the new axes are supposed to remain parallel to the primitive ones, then the formulas become, in this case,

$$x = a + x',$$

$$y = b + y',$$

$$z = c + z'.$$

321. Prop. 3. The equation to any surface being given, to find the equation to the curve formed by its intersection with a plane.

Let the equation to the surface be $f(x, y, z) = 0 \ldots .$ (1),

and the plane of $x'y'$ the intersecting plane.

Suppose the curve which results from this intersection to be referred to the plane $x'y'$, its trace upon xy being assumed as the axis of x', and a line at right angles to x', in the plane $x'y'$ as the axis of y'. Then z' will $= 0$.

Hence, the formulas in Art. 319, become, in this case,

$$x = x' \cos t, x + y' \sin t, x \cos x'y', xy,$$

$$y = x' \sin t, x - y' \cos t, x \cos x'y', xy,$$

$$z = y' \sin x'y'; xy.$$

values of x, y, z being substituted in (1), the resulting eq will express the nature of the curve required.

. In the actual application of these formulas, it will be ient to denote the angles involved in them by single shall assume, therefore,

$$\angle\, x'y', xy = \theta,$$

$$\angle\, t, x, \quad = \phi,$$

$$\angle\, t\; r' \quad = \psi.$$

322. Prop. To pass ectangular, to *polar*, co-ordinates.

Let P be any point in spac erred to A as the pole, the radius vector (r) being AP.

Then the simplest form of the polar equation to P, will be that which involves r and the angles r, x; r, y; r, z.

The angles, however, most commonly employed, are those which the projection of r upon xy forms with r, and with the axis of x.

Hence, if conformably to the notation used in Chap. viii. r_z denote the *projected radius vector*, the polar co-ordinates will be

$$r, \text{ and the angles } r_z, r;\ \ r_z, x.$$

Let x, y, z be the rectangular co-ordinates of P, namely,

AM, MN, NP, (fig. 76.), and join AN, AP.

Then it is evident, that $AN = r_z$, and that AMN and PAN are right angles.

Hence, $AM = AN \cos MAN = AP \cos PAN \cos MAN$,

$$\text{or } x = r \cos r_z, r \cos r_z, x \ldots\ldots (1).$$

$$\text{Similarly } y = r \cos r_z, r \sin r_z, x \ldots\ldots (2),$$

$$z = r \sin r_z, r_z \ldots\ldots\ldots\ldots (3),$$

which values of x, y, z, when substituted in any equation between these quantities, will furnish the polar equation required.

If the pole be transferred to any point $S\ (\alpha, \beta, \gamma)$, we have only to annex α, β and γ respectively to the values of x, y and z just deduced.

Observation. If the angle r_s, x be supposed to vary from 0 to 360^0, and the angle r_s, r from 0 to $\pm\ 90^0$, then the position of each point in space will be completely determined by the signs of the trigonometrical functions of these angles. The sign, therefore, of the radius vector must always, as in Part I, be considered positive.

SECTION II.

ON THE SPHERE, CONE, AND CYLINDER; AND ON SURFACES OF REVOLUTION.

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CHAP. I.

ON THE SPHERE.

323. **Prop. 1.** To find the equation to the sphere.

The equation sought will be obtained by expressing analytically that the distance between the centre and any point on the surface of the sphere, is invariable.

Let a sphere, whose radius is (r), be referred to any system of oblique axes; suppose x', y', z' to be the co-ordinates of the centre, and x, y, z those of any point on the surface.

Then, the distance between the two points (x', y', z') and (x, y, z) is expressed by the equation

$$\begin{aligned}&(x-x')^2+(y-y')^2+(z-z')^2\\&+2(x-x')(y-y')\cos x, y+2(x-x')(z-z')\cos x, z\\&+2(y-y')(z-z')\cos y, z=r^2,\end{aligned}$$

which is the equation required.

Cor. When the origin is at the centre, then x', y', z' are each $=0$, and the equation becomes

$$x^2+y^2+z^2+2xy\cos x,y+2xz\cos x,z+2yz\cos y,z=r^2.$$

324. **Prop. 2.** To find the various forms which the equation to the sphere assumes, when the axes are supposed to be rectangular.

Since $\cos x,y$, $\cos x,z$, and $\cos y,z$ each $=0$, the general equation becomes

$$(x-x')^2+(y-y')^2+(z-z')^2=r^2\ldots\ldots(1).$$

Let the origin be upon the sphere.

Then, since $x'^2+y'^2+z'^2=r^2$, equation (1) becomes

$$x^2-2xx'+y^2-2yy'+z^2-2zz'=0\ldots\ldots(2).$$

Let the origin be upon one of the co-ordinate planes, upon that of xy, for example.

Then, since $z'=0$, equation (1) becomes

$$(x-x')^2+(y-y')^2+z^2=r^2\ldots\ldots\ldots\ldots(3).$$

Let the origin be upon one of the axes, upon that of z, for example.

Then, since y' and z' each $=0$, equation (1) becomes

$$(x-x')^2+y^2+z^2=r^2\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

When the origin is supposed to be upon either of the planes xz, yz, or upon either of the axes of y, z, the form of the equation will be similar to that of (3) or of (4).

Let the origin be at the centre.

Then, since x', y' and z' each $=0$, equation (1) becomes

$$x^2+y^2+z^2=r^2\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Observation. The equations most generally used are those marked (1) and (5).

325. **Prop. 3.** To find the intersection of the sphere with each of the co-ordinate planes.

Let the sphere intersect the plane of xy.

Then, since $z=0$, equation (1), in the last article, becomes

$$(x-x')^2+(y-y')^2+z'^2=r^2;$$
$$\therefore\ (x-x')^2+(y-y')^2=r^2-z'^2,$$

which is the equation to a circle, the co-ordinates of whose centre are x' and y', and whose radius $=\sqrt{(r^2-z'^2)}$.

In like manner it may be shewn that the sections of the sphere, made by the planes of xz and yz, are circles.

326. **Prop. 4.** To find the section of the sphere made by any plane whatever.

Let the sphere be referred to a system of rectangular co-ordinates originating at the centre; then, its equation is

$$x^2+y^2+z^2=r^2\ldots\ldots\ldots\ldots(1).$$

The intersecting plane being supposed to pass through a point (a, b, c), the nature of the section will be determined by substituting for x, y, z, in (1), the following values (321)

$$x=a+x'\cos\phi+y'\sin\phi\cos\theta,$$
$$y=b+x'\sin\phi-y'\cos\phi\cos\theta,$$
$$z=c+y'\sin\theta.$$

This operation being executed, and the resulting expression reduced, we have

$$y'^2+x'^2+2\{(a\sin\phi-b\cos\phi)\cos\theta+c\sin\theta\}y'+2(a\cos\phi+b\sin\phi)x'$$
$$=r^2-\{a^2+b^2+c^2\},$$

which is the equation to a circle.

Whence it appears that every section of a sphere, made by a plane, is a circle.

327. Prop. 5. To find the equation to a tangent plane drawn through a given point (x', y', z') of the sphere.

The equation to a sphere, when referred to rectangular axes originating at the centre, is

$$x^2+y^2+z^2=r^2 \ldots\ldots\ldots\ldots\ldots(1).$$

Also, the equation to a plane drawn through the given point (x', y', z'), is

$$A(x-x')+B(y-y')+C(z-z')=0 \ldots(2),$$

in which A, B, C, are to be determined.

When this plane is supposed to touch the sphere, it will be at right angles to the radius passing through the point of contact; now, the equations to the radius being

$$x=\frac{x'}{z'}z; \quad y=\frac{y'}{z'}z,$$

we have (264) $A=\frac{x'}{z'}C$, and $B=\frac{y'}{z'}C$;

therefore, substituting these values of A and B in (2), and dividing by C, there results

$$x'(x-x')+y'(y-y')+z'(z-z')=0,$$

and, multiplying out, and transposing, we have

$$xx'+yy'+zz'=x'^2+y'^2+z'^2$$

$$=\therefore r^2,$$

which is the equation required.

The same equation may also be thus obtained.

The plane

$$x \cos \rho, x + y \cos \rho, y + z \cos \rho, z = \rho \text{.........}(1),$$

will evidently touch the sphere when $\rho = r$.

But, in this case, $\cos \rho, x = \cos r, x = \frac{x'}{r}$,

$$\cos \rho, y = \cos r, y = \frac{y'}{r},$$

$$\cos \rho, z = \cos r, z = \frac{z'}{r};$$

therefore, by substitution, equation (1) becomes

$$xx' + yy' + zz' = r^2,$$

as before.

328. Cor. It may easily be shewn that if the sphere be defined by the equation

$$(x - a)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2,$$

the equation to the tangent plane, applied at the point (x', y', z'), will be

$$(x' - a)(x - a) + (y' - \beta)(y - \beta) + (z' - \gamma)(z - \gamma) = r^2.$$

329. Prop. 6. To find the equation to a plane, which is a tangent to *two* given spheres.

The axes being assumed rectangular, we shall suppose, for the sake of simplicity, that the plane of xy passes through the centres of the two spheres, and that the axis of x coincides with the line joining their centres.

Hence, if r, r' be the radii of the spheres, and δ the distance between their centres, the equations to the spheres will be

$$x'^2 + y'^2 + z'^2 = r^2 \dots\dots\dots\dots (1),$$

$$(x'' - \delta)^2 + y''^2 + z''^2 = r'^2 \dots\dots\dots (2).$$

Now the equation to the plane which touches the first sphere is (327)

$$xx' + yy' + zz' = r^2 \dots\dots\dots (3),$$

and, in order that it may also touch the second sphere, the perpendicular dropped upon it from the centre of the latter must be equal to r'.

The length of this perpendicular

$$= \pm \frac{\delta x' - r^2}{\sqrt{(x'^2 + y'^2 + z'^2)}} \quad (264),$$

and, as the spheres are situated between the tangent plane and the plane of xy, the lower sign must be used;

$$\therefore\ r' = - \frac{\delta x' - r^2}{\sqrt{(x'^2 + y'^2 + x'^2)}} = \therefore\ - \frac{\delta x' - r^2}{r};$$

$$\therefore\ x' = \frac{r}{\delta}(r - r');$$

whence, substituting this value of x' in (3), and transposing,

$$zz' = r^2 - yy' - \frac{r}{\delta}(r - r')\,x,$$

$$\text{but } z' = \sqrt{(r^2 - x'^2 - y'^2)};$$

$$\therefore\ z = \frac{\delta r^2 - \delta y' y - r(r - r')\,x}{\sqrt{(r^2 - x'^2 - y'^2)}},$$

or replacing x'^2 by its value above deduced,

$$z = \frac{\delta r^2 - \delta y' y - r(r - r')\,x}{\sqrt{\{\delta^2 r^2 - r^2 (r - r')^2 - \delta^2 y'^2\}}},$$

which is the equation required.

330. Cor. Let $z=0$, then, since

$$y' = \sqrt{(r^2 - x'^2)} = \sqrt{\left\{r^2 - \frac{r^2}{\delta^2}(r-r')^2\right\}}$$

$$= \frac{r}{\delta}\sqrt{\{\delta^2 - (r-r')^2\}},$$

we have $y = \dfrac{\delta r - (r-r')\,x}{\sqrt{\{\delta^2 - (r-r')^2\}}}$,

which is the equation to a line touching two circles.

CHAP. II.

ON THE CYLINDER.

331. *Definition.* A CYLINDRICAL surface is generated by a straight line which moves parallel to itself in space, and describes with its extremity a given curve.

The straight line is called the *generating line*, and the given curve, *the directrix*, or *base*.

The surface generated is said to be that of a *right*, or of an *oblique*, cylinder, according as the generating line is at right angles, or inclined, to the plane of the directrix.

332. PROP. 1. To find the equation to cylindrical surfaces in general.

Let the equations to the straight line in any position be

$$\left.\begin{aligned} x &= az + \alpha \\ y &= bz + \beta \end{aligned}\right\} \dots\dots\dots\dots (1),$$

the quantities α, β being variable, while a, b remain constant, since the line is always parallel to itself.

Let the directrix be supposed, for the sake of simplicity, to lie in the plane of xy, and let its equation, therefore, be

$$f(X, Y) = 0 \dots\dots\dots (2).$$

Then, since the generating line always moves through a point $(x = X,\ y = Y,\ z = 0,)$ of the directrix, we have

$$X = \alpha = \therefore\ x - az,$$

$$Y = \beta = \therefore\ x - bz.$$

Hence, if these values of X and Y be substituted in the equation (2), there results

$$f\{x-az,\ y-bz\}=0,$$

which is the equation required.

333. Ex. To find the equation to an oblique cylinder, when the base is supposed to be a circle.

If the origin be at the extremity of the diameter $2r$, then

$$X^2+Y^2=2rX;$$

therefore, making the requisite substitution, we have, for the equation sought

$$(x-az)^2+(y-bz)^2=2r(x-az).$$

334. Prop. 2. To find the nature of the curve formed by the intersection of a plane with an oblique cylinder.

Let the cylinder be supposed to have a circular base, the vertex of the diameter being the origin.

As we are at liberty to give any inclination whatever to the intersecting plane, we shall suppose it to pass through the origin, and to be perpendicular to the plane of xz. Its trace upon xy will thus be the axis AY.

The formulas of Art. 321, therefore, become, since $x'x=90^\circ$,

$$x=y'\cos\theta,$$
$$y=x'$$
$$z=y'\sin\theta.$$

Hence, substituting these values of x, y, z in the equation to the cylinder

$$(x-az)^2+y^2=2r(x-az),$$

there results

$$(\cos\theta-a\sin\theta)^2y'^2+x'^2-2ry'(\cos\theta-a\sin\theta)=0,$$

which is the equation to the curve of intersection.

This curve is an ellipse, (203)

CHAP. III.

ON THE CONE.

335. **Def.** A CONICAL surface is generated by a line which passes through a given point in space, and describes with its extremity a given curve.

The given point, line and curve, are called, respectively, the *vertex**, the *generating line*, and the *directrix*, or *base*.

Since the generating line may be produced indefinitely upwards, the surface will be composed of two similar portions, one above, and the other below, the vertex; each portion is called a *sheet*†, this term being understood to bear the same relation to *surface*, that *branch* does to *curve*.

336. The conical surface, usually considered, is that whose base is a circle; the line joining the vertex and centre of the base is called the *axis* of the cone. The surface generated is said to be that of an *oblique*, or *right*, cone, according as the axis is inclined, or perpendicular, to the plane of the base.

337. **Prop. 1.** To find the equation to a conical surface in general.

Let the co-ordinates of the vertex be x', y', z'.

Then the equations to the generating line will be of the form

---

* This point is sometimes called the *centre*.

† The word used by French writers is *nappe*.

$$\left.\begin{array}{l} x-x'=a(z-z') \\ y-y'=b(z-z') \end{array}\right\} \dots\dots\dots (1).$$

the quantities a, b, being indeterminate.

Let the equation to the directrix, which is supposed as in the last Chapter to lie in the plane of xy, be

$$f(X, Y)=0 \dots\dots\dots (2).$$

Since the generating line must always pass through a point

$$(x=X,\ y=X,\ z=0),$$

of the directrix, we have

$$\left.\begin{array}{l} X-x'=-az', \text{ or } X=x'-az' \\ X-y'=-bz', \text{ or } Y=y'-bz' \end{array}\right\} \dots\dots (3);$$

therefore, substituting these values of X, Y in (2), we have

$$f\{(x'-az'),\ (y'-bz')\}=0,$$

or, since x', y', z' are given,

$$f(a, b)=0,$$

but $a=\dfrac{x-x'}{z-z'}$, $b=\dfrac{y-y'}{z-z'}$, from equation (1);

therefore, by substitution, there results

$$f\left\{\frac{x-x'}{z-z'},\ \frac{y-y'}{z-z'}\right\}=0,$$

which is the equation required.

338. Ex. 1. To find the equation to a right cone, of which the base is a circle.

Since the vertex is situated in the line which is drawn from the centre of the circular base at right angles to the plane of xy, we shall have for the equation to the directrix,

$$(X-x')^2+(Y-y')^2=r^2;$$

therefore replacing $X-x'$ and $Y-y'$ by their values in (3), we have

$$a^2z'^2+b^2z'=r^2;$$

hence, substituting for a and b, there results

$$(x-x')^2+(y-y')^2=\frac{r^2}{z'^2}(z-z')^2,$$

which is the equation required.

339. Cor. The centre of the directrix being assumed as the origin, x' and $y'=0$, and $z'=c$; the equation, therefore, becomes

$$x^2+y^2=\frac{r^2}{c^2}(z-c)^2.$$

This is the equation by which the right cone is usually characterized.

340. Ex. 2. To find the equation to a cone, of which the base is an ellipse.

The centre of the ellipse being assumed as the origin, let the equation to the directrix be

$$A^2Y^2+B^2X^2=A^2B^2 \dots\dots\dots (1).$$

Now, since x' and y' each $=0$, the values of X and Y in (3) become $-az'$ and $-bz'$, or $-z'.\frac{x}{z-z'}$; and $-z'\frac{y}{z-z'}$. We have, therefore, by substitution in (1),

$$z'^2(A^2y^2+B^2x^2)=(z-z')^2.A^2B^2,$$

$$\text{or } \frac{y^2}{B^2}+\frac{x^2}{A^2}=\left(\frac{z-z'}{z'}\right)^2,$$

which is the equation required.

341. Ex. 3. To find the equation to an oblique conical surface, when the base is supposed to be a circle.

Let the equation to the directrix be

$$(X-\alpha)^2+(Y-\beta)^2=r^2,$$

then, by substituting for X and Y their values $x'-az'$, $y'-bz'$, we have

$$(x'-az'-\alpha)^2+(y'-bz'-\beta)^2=r^2,$$

or, replacing a and b by $\frac{x-x'}{z-z'}$ and $\frac{y-y'}{z-z'}$,

$$\left\{x'-z'\frac{x-x'}{z-z'}-\alpha\right\}^2+\left\{y'-z'\frac{y-y'}{z-z'}\right\}^2=r^2,$$

which on reduction becomes

$$\{zx'-xz'-\alpha(z-z')^2\}^2+\{zy'-yz'-\beta(z-z')\}^2=r^2(z-z')^2,$$

the equation required.

342. Cor. 1. If the origin be on the circumference of the base, then

$$\alpha=r, \text{ and } \beta=0;$$

therefore the equation becomes

$$(zx'-xz')^2+(zy'-yz')^2=2r(z-z')(zx'-xz').$$

343. Cor. 2. When the origin is at the centre of the base, then α and β each $=0$, and the equation becomes

$$(zx'-xz')^2+(zy'-yz')^2=r^2(z-z')^2 \ldots\ldots (1).$$

If, moreover, the axis be in the plane of xz, then $y'=0$, and this equation assumes the form

$$(zx'-xz')^2+y^2z'^2=r^2(z-z')^2,$$

or $z'^2(x^2+y^2)+(x'^2-r^2)z^2-2x'z'.xz+2x'r^2z=z'^2r^2=0 \ldots (2)$.

When, in this equation $x'=0$, the axis is at right angles to the plane of the base, and we again fall on the equation to the right cone.

344. Prop. 2. To find the nature of the curve formed by the intersection of a *right* cone and a plane.

Let the intersecting plane be perpendicular to that of xz, and meet the axis in C, (fig. 86.) assume this point as the origin, and take $CA = c$, $CB = r$; then the equation to the cone will be (339),

$$x^2 + y^2 = \frac{r^2}{c^2}(z - c)^2.$$

The same formulas being employed as in Art. 334, we have, on making the requisite substitutions,

$$y'^2 \cos^2\theta + x'^2 = \frac{r^2}{c^2}(y' \sin\theta - c)^2;$$

whence the equation to the curve of intersection is

$$(c^2 \cos^2\theta - r^2 \sin^2\theta)\, y'^2 + c^2 x'^2 + 2cr^2 \sin\theta \,.\, y' - r^2 c^2 = 0 \;\ldots\ldots (1).$$

This equation belongs to a line of the second order, and will represent an ellipse, an hyperbola, or a parabola, according as

$$c^2 \cos^2\theta - r^2 \sin^2\theta < \text{ or } > \text{ or } = c^2;$$

that is, according as

$$\tan\theta < \text{ or } > \text{ or } = \frac{c}{r},$$

$$\text{or } \theta < \text{ or } > \text{ or } = ADC.$$

The curve will, therefore, be

(1) An ellipse, *when the intersecting plane is inclined to the generating line, and meets one sheet only of the surface.*

(2) A parabola, *when the plane is parallel to the generating line.*

(3) An hyperbola, *when the plane meets both sheets of the surface.*

Observation. The varieties of the three curves, as enumerated in Art 212, may easily be deduced from equation (1), by giving different inclinations to the intersecting plane.

345. Prop. 3. To find the nature of the curve formed by the intersection of an *oblique* cone and a plane.

Let the point C, where the intersecting plane meets the axis of the cone, be assumed as the origin; draw the plane of xy parallel to the base, and that of xz perpendicular to the given plane.

Then, if the axis of the cone be supposed to lie in the plane of xz, and the co-ordinates, therefore, of A be a and c, the equation to the conical surface will (343) be

$$c^2(x^2+y^2)+(a^2-r^2)z^2-2acxz+2ar^2z+c^2r^2=0.$$

Substituting, now, for x, y, z the same values as in the last Proposition, there results

$$c^2(y'^2\cos^2\theta+x'^2)+(a^2-r^2)y'^2\sin^2\theta-2ac\sin\theta\cos\theta\,.\,y'^2$$
$$+2ar^2y'\sin\theta-c^2r^2=0.$$

Which, becomes, when arranged,

$$\{(c\cos\theta-a\sin\theta)^2-r^2\sin^2\theta\}\,y'^2+c^2x'^2+2ar^2\sin\theta\,.\,y'-c^2r^2=0.$$

This equation belongs to a line of the second order, and will represent an ellipse, an hyperbola, or a parabola, according as

$$(c\cos\theta-a\sin\theta)^2-r^2\sin^2\theta > \text{ or } < \text{ or } =0,$$

$$\text{or } c\cos\theta-a\sin\theta < \text{ or } > \text{ or } = \pm r\sin\theta,$$

$$\text{or } \tan\theta < \text{ or } > \text{ or } = \frac{c}{a\pm r},$$

$$\text{or } \theta < \text{ or } > \text{ or } = ADB \text{ or } ABD, \text{ fig. 87.}$$

Hence, the curve will be an ellipse, an hyperbola, or a parabola, under precisely the same circumstances as in the right cone.

346. Cor. In order that the curve may be a circle, the coefficients of y'^2 and of x'^2 must be equal;

$$\therefore (c\cos\theta-a\sin\theta)^2-r^2\sin^2\theta=c^2,$$

$$a^2\sin^2\theta-(1-\cos^2\theta)\,c^2-r^2\sin^2\theta=2ac\sin\theta\cos\theta;$$

$$\therefore \tan\theta=\frac{2ac}{a^2-c^2-r^2};$$

which value of θ may easily be found.

For since $\tan ADB = -\tan ADX = -\dfrac{c}{a-r}$,

and $\tan ABD$ . . . . $\dfrac{c}{a+r}$;

$$\therefore \tan(ADB - ABD) = -\frac{2ac}{a^2 - c^2 - r^2} = -\tan\theta = \tan(\pi - \theta),$$

$$\text{or } ADB - ABD = \pi - \theta.$$

Draw DE, making the $\angle ADE = ABD$, and through C draw NCM parallel to DE; then DCM is evidently the value of θ.

The curve of intersection, therefore, will be a circle, when the angle $AMN = ADB$, or $ANM = ABD$.

This section is called the *sub-contrary section.*

Observation. It appears from the last two propositions, that the curves which result from the intersection of a cone by a plane, are *Lines of the second order;* for this reason they are generally denominated *Conic Sections.*

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CHAP. IV.

ON SURFACES OF REVOLUTION.

347. **Def.** If a curve revolve about a fixt line, or *axis*, in such a manner that each point of the curve may describe a circle whose centre is on that line, and whose plane is perpendicular to it, the surface so generated is called a *surface of revolution.*

It follows from this definition, that if the surface be cut by a plane perpendicular to the axis, the intersection will be a circle.

We may, therefore, consider the surface as formed by a circle of variable magnitude, which moves in a direction parallel to itself, and meets the generating curve.

348. **Prop. 1.** To find the equation to a surface of revolution in general.

Let the equations to the generating curve be

$$\left.\begin{array}{l} f(x, y, z) = 0 \\ f'(x, y, z) = 0 \end{array}\right\} \ldots\ldots (1).$$

Then, if x', y', z' be the co-ordinates of any point in the axis of revolution, the equations to the axis will be

$$\left.\begin{array}{l} x - x' = a(z - z') \\ y - y' = b(z - z') \end{array}\right\} \ldots\ldots\ldots (2),$$

the equation to a plane perpendicular to the axis will, therefore, be (263)

$$ax+by+z=c \ldots\ldots\ldots\ldots(3),$$

and that to a sphere, whose centre is (x', y', z'),

$$(x-x')^2+(y-y')^2+(z-z')^2=r^2 \ldots\ldots(4).$$

Now, the circle which results from the intersection of the sphere (4), by the plane (3), may be conceived to be one of those which compose the surface. Hence, the quantities c and r^2, or their equals

$$ax+by+z, \text{ and } (x-x')^2+(y-y')^2+(z-z')^2,$$

must, for the same or different points, be constant or variable together, in other words, either of them must be a function of the other. Hence

$$(x-x')^2+(y-y')^2+(z-z')^2=F(ax+by+z)$$

is the equation to surfaces of revolution in general.

349. Cor. Let the axis of revolution be supposed to coincide with any one of the co-ordinate axes, that of z for example.

Then, the equations to the *variable circle*, that is, to any one of the circles composing the surface, will be

$$z=c, \quad x^2+y^2=r^2.$$

Whence the equation to the surface becomes

$$x^2+y^2=F(z).$$

The form of the equation is similar when the axis of revolution coincides with the axis of x or of y.

Examples.

350. In the following examples, we shall suppose the generating curve to be in the plane of xz.

(1) Let the generating curve be a parabola;

$$\text{Here } y=0,\ x^2=2pz \ldots\ldots(1),$$

are the equations to the curve,

$$\text{and } z=c,\ x^2+y^2=r^2 \ldots.(2),$$

those to the variable circle.

Then, in order that this circle may pass through a point of the curve, the co-ordinates in these equations must be identical;

$$\text{Now, when } y=0,\ x^2=r^2,$$

$$\text{and when } z=c,\ x^2=2pc;$$

$$\therefore\ r^2=2pc$$

which is the *equation of condition.*

Hence, substituting for r^2 and c their values in (2), we have

$$x^2+y^2=2pz,$$

which is the equation required.

This surface is called a ***paraboloid.***

(2) Let the generating curve be an ellipse, and let the revolution be performed about the *minor* axis.

$$\text{Here } y=0,\ a^2z^2+b^2x^2=a^2b^2 \ldots\ldots(1),$$

are the equations to the curve,

$$\text{and } z=c,\ x^2+y^2=r^2 \ldots\ldots\ldots(2),$$

those to the variable circle.

As before, the equation is

$$a^2c^2 + b^2r^2 = a^2b^2,$$

whence $a^2z^2 + b^2(x^2 + y^2) = a^2b^2$

is the equation sought.

In like manner, when the revolution is performed about the *major* axis, the equation to the surface generated is

$$b^2x^2 + a^2(y^2 + z^2) = a^2b^2.$$

The solid generated by the revolution of an ellipse about either of its axes, is called a *spheroid;* and it is said to be *oblate* or *prolate*, according as the revolution is performed about the *minor* or the *major* axis.

The general equation to the Earth's surface, supposed to be that of an oblate spheroid, the difference of whose axes is extremely small, is

$$a^2z^2 + b^2(x^2 + y^2) = a^2b^2.$$

Assume $a = 1$, $b = 1 - \delta$; δ being extremely small, then

$$b^2 = 1 - 2\delta + \delta^2 = 1 - 2\delta, \text{ nearly};$$

therefore, by substitution, the above equation becomes

$$x^2 + y^2 + z^2 - 1 - 2\delta(x^2 + y^2 - 1) = 0.$$

(3) Let the generating curve be the hyperbola.

Then, the equation to the surface generated, or to the *hyperboloid of revolution*, is

$$b^2(x^2 + y^2) - a^2z^2 = a^2b^2.$$

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SECTION III.

SURFACES OF THE SECOND ORDER.

I.

ON SURFACES OF ORDER IN GENERAL.

351. anner as lines, are classified according uations. Thus the plane, since its equation is of gree, is called a surface of the *first order*, and the linder and cone, are, for a similar reason, called surfaces of the *second order*.

In general, if

$$ax^2+by^2+cz^2+2a'yz+2b'xz+2c'xy+2a''x+2b''y+2c''z+d=0$$

be the equation of the second degree between three variables, the surfaces, which are the loci of this equation, are said to be of the second order.

Throughout this Section, we shall suppose the co-ordinate planes to have any inclination whatever, unless the contrary be specified.

352. Prop. 1. A straight line cannot intersect a surface of the second order in more than two points.

Let the surface be cut by the line whose equations are

$$x=mz+\alpha,$$
$$y=nz+\beta.$$

Then, at the points of intersection, the co-ordinates of the line and of the surface are identical; substituting, therefore, the values of x and y in the general equation,

$$ax^2+by^2+cz^2+2a'yz+2b'xz+2c'xy+2a''x+2b''y+2c''z+d=0,$$

there will result a quadratic equation in terms of z, which can have only two roots. Whence the truth of the proposition.

353. Prop. 2. To find the locus of the middle points of any number of parallel chords.

$$\text{Let } x=mz,\quad y=nz \ldots\ldots\ldots\ldots (g)$$

be the equations to any line drawn through the origin and cutting the surface in the points P and p: take O (x', y', z'), the middle point of any chord Qq, parallel to Pp; then, the question is to find the relation between x', y', and z'.

If the origin be transferred to O, the equation to the surface will become

$$\begin{gathered} a(x+x')^2+b(y+y')^2+c(z+z')^2 \\ +2a'(y+y')(z+z')+2b'(x+x')(z+z')+2c'(x+x')(y+y') \\ +2a''x+2b''y+2c''z+d=0, \end{gathered}$$

and the equations to Qq,

$$x=mz,\quad y=nz.$$

Now, the points in which Qq cuts the surface, will be determined by supposing the co-ordinates in the respective equations identical, whence

$$\begin{gathered} a(mz+x')^2+b(nx+y')^2+c(z+z')^2 \\ +2a'(nz+y')(z+z')+2b'(mz+x')(z+z') \\ +2c'(mz+x')(nz+y')+2a''(mz+x') \\ +2b''(nz+y')+2c''(z+z')+d=0. \end{gathered}$$

SECTION III.

ON SURFACES OF THE SECOND ORDER.

CHAP. I.

ON SURFACES OF THE SECOND ORDER IN GENERAL.

351. SURFACES, in the same manner as lines, are classified according to the degree of their equations. Thus the plane, since its equation is of the first degree, is called a surface of the *first order*, and the sphere, cylinder and cone, are, for a similar reason, called surfaces of the *second order*.

In general, if

$$ax^2+by^2+cz^2+2a'yz+2b'xz+2c'xy+2a''x+2b''y+2c''z+d=0,$$

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be the equations to any line drawn through the origin and cutting the surface in the points P and p: take O (x', y', z'), the middle point of any chord Qq, parallel to Pp; then, the question is to find the relation between x', y', and z'.

If the origin be transferred to O, the equation to the surface will become

$$\begin{gathered} a(x+x')^2+b(y+y')^2+c(z+z')^2 \\ +2a'(y+y')(z+z')+2b'(x+x')(z+z')+2c'(x+x')(y+y') \\ +2a''x+2b''y+2c''z+d=0, \end{gathered}$$

and the equations to Qq,

$$x=mz, \quad y=nz.$$

Now, the points in which Qq cuts the surface, will be determined by supposing the co-ordinates in the respective equations identical, whence

$$\begin{aligned} & a(mz+x')^2+b(nx+y')^2+c(z+z')^2 \\ & +2a'(nz+y')(z+z')+2b'(mz+x')(z+z') \\ & +2c'(mz+x')(nz+y')+2a''(mz+x') \\ & +2b''(nz+y')+2c''(z+z')+d=0. \end{aligned}$$

But, Qq being bisected in O, the two values of z in this equation will be equal, and the coefficient of the second term will, therefore, vanish; whence, collecting the terms involving z, we have

$$2amx'+2bny'+2cz'+2a'nz'+2a'y'$$
$$+2b'mz'+2b'x'+2c'nx'+2c'ny'$$
$$+2a''m+2b''n+2c''=0,$$

or, arranging, dividing each term by 2, and suppressing the accents of the variables,

$$(am+c'n+b')x+(c'm+bn+a')y+(b'm+a'n+c)z+a''m+b''n+c''=0\ ..($$

the equation to *a plane*, which is, therefore, the locus required.

This plane, from its property of bisecting a system of parallel chords, is called *a diametral plane*.

354. Cor. In like manner, if there be two other chords,

$$x=m'z+\alpha';\quad y=n'z+\beta'\ \ldots\ldots(h),$$
$$x=m''z+\alpha'';\quad y=n''z+\beta''\ \ldots\ldots(k),$$

the corresponding diametral plane will be

$$(am'+c'n'+b')x+(c'm'+bn'+a')y+(b'm'+a'n'+c)z+a''m'+b''n'+c''=0\ ..$$
$$(am''+c'n''+b')x+(c'm''+bn''+a')y+(b'm''+a'n''+c)z+a''m''+b''n''+c''=0.$$

355. Of the three planes marked (1), (2), and (3), the direction of the first depends upon that of the chord (g), which was assumed at pleasure. We shall now fix the relative position of the two other chords (h) and (k), by supposing, first, that each of them is parallel to (1), and next, that either of

them, *k*, for example, is parallel to the diametral plane (2) of the other.

Then, since *h* and *k* are each parallel to (1), we have (269)

$$m' \{am+c'n+b'\}+n' \{c'm+bn+a'\}+b'm+a'n+c=0,$$

$$m'' \{am+c'n+b'\}+n'' \{c'm+bn+a'\}+b'm+a'n+c=0,$$

or, arranging these two expressions,

$$b'(m+m')+a'(n+n')+c'(mn'+nm')+amm'+bnn'+c=0\,..(4),$$

$$b'(m+m'')+a'(n+n'')+c'(mn''+nm'')+amm''+bnn''+c=0\,..(5),$$

and, since (*k*) is also parallel to (2), we have, in like manner,

$$b'(m'+m'')+a'(n'+n'')+c'(m'n''+n'm'')+am'm''+bn'n''+c=0\,..(6).$$

But, because (*k*) is parallel to (1) and (2), it must be parallel to the line of their intersection, therefore the diametral plane (3) of (*k*), bisects all chords parallel to the intersection of the remaining two diametral planes.

In like manner, since (4), (5), and (6), are equations of symmetrical form, the diametral planes of (*g*) and of (*h*) will bisect the chords which are parallel to the intersection of the planes (2) and (3), and (1) and (3).

Hence, it appears that each of these three diametral planes bisects the chords which are parallel to the intersection of the other two.

Diametral planes, thus related, are said to be *conjugate to one another;* and the intersections of each two of them are called *conjugate diameters.*

The number of systems of such planes is evidently unlimited.

356. Prop. 3. To find whether any system of conjugate diametral planes can be rectangular.

For the purpose of shortening the investigation, we shall refer the surface to *rectangular* axes.

In order that any three diametral planes may be conjugate to one another, the following equations must obtain,

$$\left.\begin{array}{l} b'(m+m')+a'(n+n')+c'(mn'+nm')+amm'+bnn'+c=0 \\ b'(m'+m'')+a'(n'+n'')+c'(m'n''+n'm'')+am'm''+bn'n''+c=0 \\ b'(m+m'')+a'(n+n'')+c'(mn''+nm'')+amm''+bnn''+c=0 \end{array}\right\}\cdots(1$$

If we suppose that the conjugate diametral planes are rectangular, we shall have (251)

$$\left.\begin{array}{l} 1+mm'+nn'=0 \\ 1+m'm''+n'n''=0 \\ 1+mm''+nn''=0 \end{array}\right\}\cdots\cdots(2).$$

Our object now is to derive, by elimination from equations (1) and (2), the values of m and n; of m' and n', and of m'' and n''.

Multiplying the first of equations (1) by n'', and the last by n', and taking the difference of the products, we have

$$\{c+b'm+a'n\}\,(n''-n')+\{a'+c'm+bn\}\,(m'n''-n'm'')=0.$$

The same operation being performed on the first and last of equations (2), there results

$$n''-n'+m\,(m'n''-n'm'')=0.$$

therefore eliminating $n''-n'$ from the last two equations, we have

$$m=\frac{b'+am+c'n}{c+b'm+a'n}\cdots\cdots\cdots(3).$$

In like manner, if the first and last of equations (1) and (2) be multiplied successively by m'', m', and the difference of their respective products taken, we shall have

$$n = \frac{a' + c'm + bn}{c + b'm + a'n} \;\ldots\ldots\ldots\; (4).$$

From (3) is derived

$$n = \frac{m(a - c) + b'(1 - m^2)}{a'm - c} \;\ldots\; (5),$$

and from (4),

$$a'n^2 + \{b'm + c - b\}\, n - (c'm + a') = 0.$$

Substituting in this the value of n from (5), we arrive at a cubic equation that must contain at least *one real value of* m; to which there corresponds a real value of n, deduced from (3). In like manner, it is demonstrable that there must exist at least one real value of m' and n', and of m'' and n''.

Now, the cubic equations which serve to determine m, m', and m'', will be found identical, a circumstance which might be inferred *a priori* from the symmetrical form of equations (1) and (2): consequently, m, m', m'' are the three roots of the same cubic equation. Hence, there can be only one system of conjugate diametral planes that are rectangular.

The intersections of each two are called the *principal diameters*, and the points in which these cut the surfaces are called the *vertices*.

357 Prop. 4. To find the co-ordinates of the point in which any three diametral planes intersect one another.

Resuming equations (1), (2), and (3), and arranging them with reference to m and n; m' and n'; m'' and n'', we shall have

$$\{ax + c'y + b'z + a''\}\, m + \{c'x + by + a'z + b''\}\, n + b'x + a'y + cz + c'' = 0,$$

$$\{ax + c'y + b'z + a''\}\, m' + \{c'x + by + a'z + b''\}\, n' + b'x + a'y + cz + c'' = 0,$$

$$\{ax + c'y + b'z + a''\}\, m'' + \{c'x + by + a'z + b''\}\, n'' + b'x + a'y + cz + c'' = 0.$$

SECTION III.

ON SURFACES OF THE SECOND ORDER.

CHAP. I.

ON SURFACES OF THE SECOND ORDER IN GENERAL.

351. SURFACES, in the same manner as lines, are classified according to the degree of their equations. Thus the plane, since its equation is of the first degree, is called a surface of the *first order*, and the sphere, cylinder and cone, are, for a similar reason, called surfaces of the *second order*.

In general, if

$$ax^2+by^2+cz^2+2a'yz+2b'xz+2c'xy+2a''x+2b''y+2c''z+d=0,$$

be the equation of the second degree between three variables, the surfaces, which are the loci of this equation, are said to be of the second order.

Throughout this Section, we shall suppose the co-ordinate planes to have any inclination whatever, unless the contrary be specified.

352. PROP. 1. A straight line cannot intersect a surface of the second order in more than two points.

Let the surface be cut by the line whose equations are

$$x=mz+\alpha,$$

$$y=nz+\beta.$$

CHAP. II.

ON SURFACES WHICH HAVE A CENTRE.

359. **Prop. 1.** To find the form of the equation to surfaces of the first class, when the co-ordinate planes are parallel respectively to any system of conjugate diametral planes.

The equation to any diametral plane is (341)

$$(am+c'n+b')\,x+(c'm+bn+a')\,y+(b'm+a'n+c)\,z+a''m+b''n+c''=0.$$

If m and n, successively, be supposed infinitely great, and then each be supposed $=0$, the resulting equations, identical with (1′), (2′) and (3′) of Art. 345, will represent the diametral planes which bisect the chords parallel to the axes of x, of y and of z.

Hence, when the co-ordinate planes are parallel to any system of conjugate diametral planes, of the three equations

$$ax+c'y+b'z+a''=0,$$
$$c'x+by+a'z+b''=0,$$
$$b'x+a'y+cz+c''=0,$$

the first ought only to involve x, the second y, and the third z. It follows, therefore, that

$$a'=0,\ b'=0,\ c'=0.$$

Whence the general equation becomes

$$ax^2+by^2+cz^2+2a''x+2b''y+2c''z+d=0,$$

which is of the form required.

360. Cor. 1. To find the form of the equation, when the centre is the origin.

In this case, the equations to the three conjugate diametral planes, will assume the form which is specified in Art. 253.;

$$\therefore\ a''=0,\ b''=0,\ c''=0,$$

the general equation becomes, therefore,

$$ax^2+by^2+cz^2+2a'yz+2b'xz+2c'xy+d=0.$$

361. Cor. 2. It follows from the last two articles, that the equations to surfaces of the first class, when referred to any system of conjugate diametral planes, is of the form

$$Ax^2+By^2+Cz^2+D=0.$$

362. Prop. 2. To find the equation to surfaces of the second order, in terms of a given system of conjugate diameters.

The form of the equation to the surface, when referred to conjugate diametral planes, is (353),

$$Ax^2+By^2+Cz^2+D=0\ldots\ldots(1).$$

Let $2a'$, $2b'$, $2c'$ be the given system of conjugate diameters. Then, if in (1) we suppose x and y, x and z, y and z, successively $=0$, the resulting values of z, y, x will be respectively c', b', a'. We thus have

$$z^2 \text{ or } c'^2 = -\frac{D}{C};\ \therefore\ C = -\frac{D}{c'^2},$$

$$y^2 \text{ or } b'^2 = -\frac{D}{B};\ \therefore\ B = -\frac{D}{b'^2},$$

$$x^2 \text{ or } a'^2 = -\frac{D}{A};\ \therefore\ A = -\frac{D}{a'^2}.$$

Substituting in (1), these values of A, B, C, and dividing each term of the result by $-D$, we obtain

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1,$$

which is the equation sought.

363. **Cor.** If $2a$, $2b$, $2c$ denote the principal diameters, then the equation to the surface, when referred to these, becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Note. The quantities $2a$, $2b$, $2c$, and $2a'$, $2b'$, $2c'$ will, in future, be employed in the sense annexed to them in the last two Articles; and must, therefore, be distinguished from the coefficients of the general equation, which were denoted by the same letters.

364. **Prop. 3.** To find the equation to a tangent plane, applied at a given point (x', y', z').

The surface being referred to conjugate diametral planes, its equation is

$$Ax^2+By^2+Cz^2+D=0 \; \ldots\ldots \; (1).$$

Now, if a secant be drawn through the given point, its equations will be

$$\left.\begin{array}{l} x-x'=m\,(z-z') \\ y-y'=n\,(z-z') \end{array}\right\} \ldots\ldots \; (2).$$

But, since x', y', z' must satisfy equation (1); we have

$$Ax'^2+By'^2+Cz'^2+D=0,$$

whence, eliminating D by means of equation (1),

$$A\,(x^2-x'^2)+B\,(y^2-y'^2)+C\,(z^2-z'^2)=0,$$

$$\text{but } (x^2-x'^2)=(x+x')\,(x-x')$$
$$= \therefore (x+x')\,m\,(z-z').$$

In like manner,

$$y^2-y'^2= \quad (y+y')\,n\,(z-z').$$

Substituting, therefore, and dividing each term by $z-z'$, there results

$$Am\,(x+x')+Bn\,(y+y')+C\,(z+z)=0 \; \ldots\ldots \; (3).$$

Let us now suppose $x=x'$, $y=y'$ and $z=z'$; then the secant becomes a tangent, and we have, on dividing each term by 2,

$$Amx'+Bny'+Cz'=0.$$

But the tangent plane at a given point of a surface, being the plane which contains all the *linear* tangents at the same point*, this equation ought to be independent of m and n; eliminating these quantities, therefore, by means of (2), we have

$$Ax'.\frac{x-x'}{z-z'}+By'.\frac{y-y'}{z-z'}+Cz'=0,$$

$$\text{or } Axx'+Byy'+Czz'=Ax'^2+By'^2+Cz'^2$$
$$=\therefore -D,$$

which is the equation required.

By a similar process, the equation to the tangent plane may be determined, when the surface is defined by its general equation.

365. Cor. 1. When the surface is referred to a given system of conjugate diameters, the equation to the tangent plane will become (354)

$$\frac{xx'}{a'^2}+\frac{yy'}{b'^2}+\frac{zz'}{c'^2}=1.$$

366. Cor. 2. The equations to the normal are, (264),

$$x-x'=\frac{Ax'}{Cz'}(z-z'),$$

$$y-y'=\frac{By'}{Cz'}(z-z').$$

367. Cor. 3. The tangent plane at the extremity of any diameter is parallel to the diametral plane, which is conjugate to that diameter.

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* Lacroix, *Compl. de Geom.* No. 103.

Hence, the principal diameters are perpendicular to the tangent planes at their extremity.

368. Prop. 4. A system of conjugate diameters of known inclination being given, to find the magnitude and direction of the principal diameters.

The proposed question is analogous to that which forms the subject of Art. 119, and may be resolved on similar principles.

Any one of the principal diameters being perpendicular to the tangent plane at its extremity (359), we shall assume as the basis of the investigation, that this plane coincides with the tangent plane applied at the same point to a concentric sphere.

Let x', y', z' be the co-ordinates of the extremity of any principal diameter, and $2r$ its length.

The equations to the diameter will be

$$x = mz, \quad y = nz \;\ldots\ldots\ldots\; (1),$$

$$\text{in which } m = \frac{x'}{z'}, \quad \text{and } n = \frac{y'}{z'}.$$

The equation to the plane which touches the surface at the extremity of this diameter is, (357),

$$\frac{x'}{a'^2}x + \frac{y'}{b'^2}y + \frac{z'}{c'^2}z = 1,$$

or replacing $\frac{x'}{z'}$ and $\frac{y'}{z'}$ by their values,

$$\frac{m}{a'^2}x + \frac{n}{b'^2}y + \frac{1}{c'^2}z = 1 \;\ldots\ldots\; (2).$$

In like manner, the equation to the tangent plane to a concentric sphere, at the same point, may be shewn to be

$$\frac{1}{r^2}\{m+n\cos x,y+\cos x,z\}\,x+\frac{1}{r^2}\{n+m\cos x,y+\cos y,z\}\,y$$

$$+\frac{1}{r^2}\{1+n\cos y,z+m\cos x,z\}\,z=1\;.\;.\;.\;.\;.\;(3).$$

Now, since these tangent planes are identical, we have, equating their corresponding terms,

$$m+n\cos x,y+\cos x,z=\frac{m}{a'^2}r^2,$$

$$n+m\cos x,y+\cos y,z=\frac{n}{b'^2}r^2,$$

$$1+n\cos y,z+m\cos x,z=\frac{1}{c'^2}r^2.$$

Whence, eliminating m and n, we obtain the equation

$$r^6-(a'^2+b'^2+c'^2)\,r^4+\{a'^2b'^2\sin^2 x,y+a'^2c'^2\sin^2 x,z+b'^2c'^2\sin y,z\}\,r^2$$
$$-a'^2b'^2c'^2\{1-2\cos x,y\cos x,z\cos y,z-(\cos^2 x,y+\cos^2 x,z+\cos^2 y,z)\}=0,$$

the roots of which are evidently real.

Hence, the magnitude of the principal diameters is known, and their direction, dependent on m and n, may also be determined.

369. COR. Let a^2, b^2, c^2, arranged in the order of their magnitudes, be the three absolute values of r^6, (355).

We then have, by the theory of equations,

$$(1)\quad a'^2+b'^2+c'^2=a^2+b^2+c^2,$$

or, *the sum of the squares of any system of conjugate diameters is equal to the sum of the squares of the principal diameters.*

$$(2)\quad a'^2b'^2\sin^2 x,y+a'^2c'^2\sin^2 x,z+b'^2c'^2\sin^2 y,z=a^2b^2+a^2c^2+b^2c^2,$$

or, *the sum of the squares of the faces of the parallelepiped described on any system of conjugate diameters, is equal to the* *of the squares of the faces of the rectangular parallelepiped scribed on the principal diameters.*

$$\therefore\ a'^2 b'^2 c'^2 \{1 - 2\cos x,y \cos x,z \cos y,z - (\cos^2 x,y + \cos^2 x,z + \cos^2 y,z)\}$$
$$= a^2 b^2 c^2.$$

r, the volume of the parallelepiped, on any system of conjugate diameters, is equal to that on the principal diameters.

When any of the quantities a, a'; b, b'; c, c' are imaginary, their squares must be reckoned negative.

370. Hitherto we have paid no regard to the signs of a^2, b^2, c^2, and have employed the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ \ldots\ldots\ (1),$$

to characterize, indiscriminately, all surfaces of the second order that have a centre. But, since a^2, b^2, c^2 are the three roots of a cubic equation, they admit of the following varieties of sign:

(1) They may be all positive.

(2) Two may be positive, and one negative.

(3) One may be positive, and two negative.

(4) They may be all negative.

Wherefore, equation (1) will undergo a corresponding change, and will, in each of the first three cases, belong to a particular species of surface; the fourth case we exclude from consideration, because the surface would then be imaginary.

It thence appears, that surfaces of the first class comprehend three species; these are named as follows:

(1) The *Ellipsoid.*

Here a^2, b^2, c^2 are positive; the equation, therefore, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Hyperboloid of *one sheet.*

are positive, and c^2 negative; the equation, there-

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

(3) *Hyperboloid* of *two sheets.*

Here positive, and b^2, c^2 negative; the equation, therefore, is

$$= 1.$$

We now nature and form of each of these su

CHAP. III.

ON THE ELLIPSOID.

371. **Prop. 1.** To find the nature of the section made by a plane parallel to any one of the co-ordinate planes.

If the intersecting plane be supposed parallel to the plane of xy, its equation will (256) be $z = \gamma$.

This value of z being substituted in the equation to the ellipsoid, there results

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{\gamma^2}{c^2} = 1,$$

$$\text{or } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{\gamma^2}{c^2},$$

which, so long as $\gamma < c$, is the equation to an ellipse.

The sections made by planes parallel to the planes of xz and yz, may, in like manner, be proved to be ellipses whose equations are respectively

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 - \frac{\beta^2}{b^2},$$

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{\alpha^2}{a^2},$$

372. Cor. The *traces* of the ellipsoid, that is, its intersections with the co-ordinate planes, are determined by making x, y, z, successively $= 0$ in the general equation, or, which amounts to the same thing, by making α, β, γ in the last article $= 0$. The equations thus obtained are

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ the trace on } yz \ldots\ldots\ldots (1),$$

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots\ldots xz \ldots\ldots\ldots (2),$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots\ldots\ldots\ldots xy \ldots\ldots\ldots (3),$$

whence, the traces are ellipses whose axes are $2b$, $2c$; $2a$, $2c$; and $2a$, $2b$.

These traces are called the *principal sections* of the ellipsoid, and may be thus represented, fig. 88.

Let the centre C be the origin, then

(1) The section made by the plane of xy, is the ellipse $ABVb$, whose equation is (1), and whose axes, therefore, are $2a$, $2b$.

(2) The section made by the plane of xz, is the ellipse $ADVd$, whose equation is (2), and whose axes, therefore, are $2a$, $2c$.

(3) The section made by the plane of yz, is the ellipse $DBdb$, whose equation is (3), and whose axes are $2b$, $2c$.

The origin is the common centre of these three ellipses, and the lines which are marked by the same letters must be conceived to coincide.

The six vertices are A, V, B, b, D, d.

373. Prop. 2. To find whether or not the ellipsoid be limited in all directions.

In order to ascertain this circumstance, we must determine the co-ordinates of the points in which any diameter intersects the surface. For, according as these co-ordinates are real or imaginary, the surface will be limited or unlimited, in the direction indicated by each.

Let the equations, then, to any diameter be

$$x = mz, \quad y = nz.$$

Substituting these values of x and y, in the equation to the ellipsoid, we have

$$z = \pm \frac{abc}{\sqrt{\{b^2c^2m^2 + a^2c^2n^2 + a^2b^2\}}},$$

whence x and y are also known.

Now, whatever values be assigned to m and n, the denominator of this fraction will continue finite, and z, therefore, will always be real: consequently, x and y will also be real. Hence, every diameter meets the surface; which will, therefore, consist of *one sheet*, and be limited in all directions.

374. The general equation to the ellipsoid being

$$Ax^2 + By^2 + Cz^2 = D,$$

in which A, B, C, are positive, let us suppose

(1) That $D = 0$.

Then the equation becomes

$$Ax^2 + By^2 + Cz^2 = 0,$$

which is the equation to a point.

(2) That D is negative.

Then, since the sum of three quantities essentially positive cannot be negative, the equation in this case is that to an imaginary surface.

(3) That any two of the coefficients A, B, C, are equal.

, then, the equation becomes

$$x^2+y^2+\frac{C}{A}.z^2=\frac{D}{A},$$

or $x^2+y^2 = F(z)$,

which (349) is the equation to an ellipsoid *of revolution*, about the axis of z.

In like manner, according as the coefficients of x^2 and z^2, or of y^2 and z^2 are equ oid becomes a surface of revolution about the axis of y or respectively.

(4) That the B, C, are equal.

Then, the equation

$$x^2+ \quad ,$$

It f eties of the ellipsoid are *the ellipsoid of revolution*, the *sphere*, a *point*, and an *imaginary surface*.

CHAP. IV.

ON THE HYPERBOLOID OF ONE SHEET.

375. **Prop. 1.** To find the nature of the section made by a plane parallel to any one of the co-ordinate planes.

If the intersecting plane be supposed parallel to the plane of xy, its equation will be $z=\gamma$.

This value of z being substituted in the equation to the hyperboloid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

there results

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{\gamma^2}{c^2}=1,$$

$$\text{or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{\gamma^2}{c^2},$$

which is the equation to an ellipse.

Since γ^2 remains positive, whether γ be positive or negative, it follows that if planes be drawn parallel to xy, at any distance above or below it, their intersections with the surface are all ellipses.

Next, if $y=\beta$ be the equation to a plane drawn parallel to xz, the section is represented by the equation

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{\beta^2}{b^2},$$

and is, therefore, an hyperbola.

In like manner, the section made by a plane parallel to y is an hyperbola, whose equation is

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{\alpha^2}{a^2}.$$

This surface will consist of only one sheet.

376. Cor. Hence, the *principal sections* of the hyperboloid are

an ellipse, whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots\ldots (1)$,

an hyperbola .... $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \ldots\ldots (2)$,

an hyperbola .... $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \ldots\ldots (3)$.

Let $x = 0$, then $y = \pm b$, and z is imaginary,

$y = 0$, $x = \pm a$, and z is imaginary.

Consequently, the axis of z does not meet the surface, which has, therefore, only four vertices.

The principal section may be thus represented, fig. 89.

The centre C being the origin,

(1) The section made by the plane of xy is an ellipse $ABVb$, whose equation is (1), and whose axes, therefore, are $2a$, $2b$.

(2) The section made by the plane of xz is an hyperbola, whose equation is (2), and whose axes are $2a$, $2c\sqrt{-1}$.

(3) The section made by the plane of yz is an hyperbola, whose equation is (1), and whose axes are $2b$, $2c\sqrt{-1}$.

The origin is the common centre of these three sections, and the lines which are marked by the same letters must be conceived to coincide.

The four vertices are A, V, B, b.

377. Prop. 2. To determine the co-ordinates of the points in which any diameter intersects the hyperboloid.

Let the equations to any diameter be

$$z = mz, \quad y = nz.$$

Substituting these values of x and y in the equation to the hyperboloid, we have

$$z = \frac{\pm abc}{\sqrt{\{b^2c^2m^2 + a^2c^2n^2 - a^2b^2\}}},$$

whence x and y are also known.

The diameter will, therefore, meet the surface only so long as the denominator $b^2c^2m^2 + a^2c^2n^2 - a^2b^2$ continues finite.

If this quantity $= 0$, then the diameter will not meet the surface, except at an infinite distance. In this case,

$$n = \pm \frac{b}{ac}\sqrt{(a^2 - m^2c^2)},$$

which will be real if $m < \frac{a}{c}$.

Hence, the equations to the *linear asymptote*, (See Art. 103.) are

$$x = mz, \quad y = \frac{b}{ac}\sqrt{(a^2 - m^2c^2)}\, z.$$

Eliminating m from these equations, we obtain

$$b^2c^2x^2 + a^2c^2y^2 - a^2b^2z^2 = 0,$$

$$\text{or } \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0,$$

which (337) is the equation to a *conical surface.*

This conical surface, containing all the linear asymptotes that can be drawn through the origin, will itself be an asymptote to the hyperboloid.

378. The general equation to the hyperboloid of one sheet being

$$Ax^2 + By^2 - Cz^2 = D,$$

let us suppose

(1) That $D = 0$.

$$\text{Then } \frac{x^2}{y^2} = \frac{D}{A}\frac{z^2}{y^2} - \frac{B}{A};$$

$$\therefore F\left(\frac{x}{z}, \frac{y}{z}\right) = 0.$$

Hence, (337) the surface becomes that of a cone, whose vertex is at the origin.

(2) That $A = B$.

Then, the equation becomes

$$x^2 + y^2 - \frac{C}{A}z^2 = \frac{D}{A},$$

$$\text{or } x^2 + y^2 = F(z).$$

The hyperboloid, therefore, becomes a surface of revolution (349) about the axis of z.

It thence appears that the varieties of the hyperboloid of one sheet are the *hyperboloid of revolution,* and *the cone.*

CHAP. V.

ON THE HYPERBOLOID OF TWO SHEETS.

379. **Prop. 1.** To find the nature of the section made by a plane parallel to any one of the co-ordinate planes.

The intersecting plane being supposed parallel to the plane of xy, its equation will be $z=\gamma$.

Substituting this value of z in the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

we have $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1+\frac{\gamma^2}{c^2},$$

which is the equation to an hyperbola.

Next, if $y=\beta$ be the equation to a plane drawn parallel to the plane of xz, the equation to the section will be

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1+\frac{\beta^2}{b^2},$$

which is also an hyperbola.

Lastly, let $x=\alpha$ be the equation to a plane drawn parallel to the plane of yz, then, the section will be defined by the equation

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{\alpha^2}{a^2}-1.$$

The section, therefore, is an *ellipse*, provided that $\frac{\alpha^2}{a^2} > 1$, or $\alpha > \pm a$.

If $\alpha = \pm a$, the section is a point,

$\alpha \angle \pm a$, the section is imaginary.

Whence it appears that if two planes be drawn parallel to the plane of xy, at a distance $+a$ and $-a$, the surface will have no point situated between these planes, but will extend indefinitely above and below them. The surface will, therefore, consist of *two* sheets.

Cor. Hence, the principal sections of the hyperboloid of two sheets, are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ \ldots\ldots\ (1),$$

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \ \ldots\ldots\ (2),$$

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = -1 \ \ldots\ldots\ (3),$$

of which the first two are hyperbolas, and the last an imaginary curve.

Let $x = 0$, then y and z are each imaginary.

Let y or $z = 0$; $\therefore x = \pm a$.

Consequently the axis of x alone meets the surface, which has, therefore, only two vertices.

The principal sections may be thus represented (fig. 90.).

The centre C being the origin,

(1) The section made by the plane of xy is an hyperbola whose equation is (1), and whose axes are $2a$, $2b\sqrt{-1}$.

(2) The section made by the plane of xz is also an hyperbola whose equation is (2), and whose axes are $2a$, $2c\sqrt{-1}$.

(3) The section made by the plane of of yz is imaginary.

The two vertices are A, V.

380. Prop. 2. To determine the co-ordinates of the points in which any diameter intersects the hyperboloid of two sheets.

It may be shewn precisely as in Art. 376, that the equations to the *linear* asymptote are

$$x = mz, \quad y = \frac{b}{ac} \sqrt{(m^2c^2 - a^2)} \, . \, z,$$

and that the equation to the conical surface, which is an asymptote to the hyperboloid, is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

381. The general equation to the hyperboloid of two sheets being

$$Ax^2 - By^2 - Cz^2 = D.$$

Suppose

(1) That $D = 0$.

Then, as in (337), the surface becomes conical.

(2) That $B = C$.

$$\text{Then } \frac{A}{B} x^2 - \frac{D}{B} = y^2 + z^2;$$

therefore, the hyperboloid becomes a surface of revolution about the axis of x.

Hence, the varieties of the hyperboloid of *two* sheets are the *hyperboloid of revolution*, and *the cone*.

The section, therefore, is

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gone the requisite modification. It was thus, when treat-
g of lines of the second order, that we derived the equation to
the parabola, from that to the ellipse or hyperbola.

383. Prop. 1. To find the equation to surfaces of the second class.

The equation to the ellipsoid, and hyperboloid of one sheet, is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=1.$$

Let the origin be transferred to the vertex of the diameter $2a$, which is done by substituting $a-x'$ for x; then the equation becomes

$$\frac{x'^2}{a^2}+\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=\frac{2x'}{a},$$

or, multiplying each term by a,

$$\frac{x'^2}{a}+\frac{a}{b^2}y^2\pm\frac{a}{c^2}z^2=2x'\ldots\ldots(1).$$

CHAP. VI.

ON SURFACES WHICH HAVE NOT A CENTRE.

382. **We** shall now direct our attention to that class of surfaces, in which the centre is supposed to be infinitely distant. The simplest method of obtaining their equation, is to deduce it from the equation to surfaces of the first class, when it has undergone the requisite modification. It was thus, when treating of *lines* of the second order, that we derived the equation to the parabola, from that to the ellipse or hyperbola.

383. Prop. 1. To find the equation to surfaces of the second class.

The equation to the ellipsoid, and hyperboloid of one sheet, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1.$$

Let the origin be transferred to the vertex of the diameter $2a$, which is done by substituting $a - x'$ for x; then the equation becomes

$$\frac{x'^2}{a^2} + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = \frac{2x'}{a},$$

or, multiplying each term by a,

$$\frac{x'^2}{a} + \frac{a}{b^2}y^2 \pm \frac{a}{c^2}z^2 = 2x' \ldots\ldots (1).$$

Now, the principal sections made by the planes of xy, xz are, in the ellipsoid, ellipses; and, in the hyperboloid, an ellipse and an hyperbola; Arts. 371, and 375.

Let m, m' denote the distances from the foci of these sections to the vertex of the diameter $2a$.

Then, supposing the centre to be infinitely distant, $\frac{x'^2}{a} = 0$, since a is infinite.

$$\text{Also } \frac{b^2}{a} = 2m, \text{ and } \frac{c^2}{a} = 2m', \text{ (158);}$$

therefore, substituting these values in (1), the equation becomes

$$\frac{y^2}{2m} \pm \frac{z^2}{2m'} = 2x',$$

or, multiplying out, and suppressing the accent of x,

$$m'y^2 \pm mz^2 = 4mm'x,$$

which is the equation sought.

The surfaces represented by this equation are called by the general name of *Paraboloid*.

They comprise the two following species:

(1) The *elliptic paraboloid*, whose equation is

$$m'y^2 + mz^2 = 4mm'x.$$

(2) The *hyperbolic paraboloid*, whose equation is

$$m'y^2 - mz^2 = 4mm'x.$$

384. Having obtained, by what appears the most obvious process, the equation to the paraboloid, we shall proceed to point out more fully the relation which the two classes of surfaces bear to each other.

Resuming the equation

$$ax^2+by^2+cz^2+2a''x+2b''y+2c''z+d=0,$$

let the origin be transposed to a point (α, β, γ); the equation then becomes

$$a(x'+\alpha)^2+b(y'+\beta)^2+c(z'+\gamma)^2$$
$$+2a''(x'+\alpha)+2b''(y+\beta)+2c''(z'+\gamma)+d=0,$$

or, when developed and arranged,

$$ax'^2+by'^2+cz'^2+(2a\alpha+a'')x'+(2b\beta+b'')y'+(2c\gamma+c'')z'$$
$$+a\alpha^2+b\beta^2+c\gamma^2+2a''\alpha+2b''\beta+2c''\gamma+d=0.$$

Now, in order that the new origin may be the centre, the coefficients of x', y', and z', must $=0$ (360);

$$\therefore\ 2a\alpha+a''=0;\quad \therefore\ \alpha=-\frac{a''}{2a},$$

$$2b\beta+b''=0;\quad \therefore\ \beta=-\frac{b''}{2b},$$

$$2c\gamma+c''=0;\quad \therefore\ \gamma=-\frac{c''}{2c}.$$

This operation is practicable only so long as the coefficients a, b, c, are finite.

When the centre is infinitely distant, we may suppose all the three quantities a, b, c, or any two of them, or only one of them, to vanish.

The first hypothesis is obviously inadmissible, because the surface would then be reduced to a plane. It only remains, therefore, to consider the last two cases.

(1) Let any two of the coefficients, a and b for example, $=0$. Then, the general equation becomes

$$cz^2+2a''x+2b''y+2c''z+d=0.$$

In order to ascertain the nature of this surface, we shall cut it by planes parallel to that of xy; substituting, therefore, γ, γ', γ''... successively for z, we have

$$2a''x+2b''y+c\gamma^2+2c''\gamma+d=0,$$

$$2a''x+2b''y+c\gamma'^2+2c''\gamma'+d=0,$$

&c.

which are the equations to *parallel straight lines;* whence (331) the surface is *cylindrical.* The species of the cylinder is determined by making $y=0$ in its equation; we thus have for the equation to its projection on the plane of xz,

$$cz^2+2a''x+2c''z+d=0,$$

which represents a *parabola.*

Hence, when the general equation wants two of the terms involving the *square* of the variables, it belongs to a cylindrical surface, whose base is a parabola.

(2) Let any one of the coefficients, for example, $a=0$.

Then, the transformed equation becomes

$$by'^2+cz'^2+2a''x'+(2b\beta+b'')y'+(2c\gamma+c'')x'$$
$$+b\beta^2+c\gamma^2+2a''\alpha+2b\beta''+...=0.$$

Here it is plain that α cannot be found, as before, by taking away the term involving x'; it may be determined however by supposing the last term to vanish. We thence have

$$\beta=-\frac{b''}{2b},\ \gamma=-\frac{c''}{2c},\text{ and } \alpha=-\frac{\beta^2+c\gamma^2+2b''\beta+2c''\gamma+d}{2a''}.$$

The equation, therefore, now becomes

$$by'^2+cz'^2+2a''x=0,$$

which is the equation to a paraboloid.

In order that the equation may be reduced to this form, the quantity a'' must be finite; whence it follows that in the equation to the paraboloid, the first and second powers of any variable cannot vanish at the same time.

If a and a'' be both supposed $= 0$ in the general equation, it becomes

$$by^2+cz^2+2b''+2c''z+d=0;$$

which represents a cylindrical surface, whose base, situated in the plane of yz, will be an ellipse or an hyperbola, according as b and c have the same or different signs.

385. **Prop. 2.** To find the equation to a tangent plane, applied at a given point (x', y', z').

The equations to a *secant* drawn through the given point is

$$\left.\begin{aligned} x-x' &= a(z-z') \\ y-y' &= b(z-z') \end{aligned}\right\} \ldots\ldots (1).$$

Since the given co-ordinates must satisfy the general equation,

$$m'y^2 \pm mz^2 = 4mm'x,$$

$$\text{we have } m'y'^2 \pm mz'^2 = 4mm'x';$$

therefore, subtracting the lower from the upper,

$$m'(y^2-y'^2) \pm m(z^2-z'^2) = 4mm'(x-x'),$$

$$\text{or } m'(y-y')(y+y') \pm m(z-z')(z+z') = 4mm'(x-x'),$$

whence, replacing $x-x'$ and $y-y'$, by their values in (1), and dividing the result by $z-z'$, we have

$$m'b(y+y') \pm m(z+z') = 4mm'a,$$

which is the equation to the secant.

Suppose, now, that $x=x'$, $y=y'$, $z=z'$, then the secant becomes a tangent, whose equation, therefore, is

$$m'by' \pm mz' = 2mm'a.$$

But, since the tangent plane contains all the linear tangents that can be applied at the given point, its equation must be independent of a and b. Wherefore, eliminating these quantities by means of (1), we have

$$m'y'\frac{y-y'}{z-z'} \pm mz' = 2mm'\frac{x-x'}{z-z'};$$

$$\therefore\ m'yy' - m'y'^2 \pm mzz' \mp mz'^2 = 2mm'x - 2mm'x',$$

$$\text{or } m'yy' \pm mzz' - 2mm'x = m'y'^2 \pm mz'^2 - 2mm'x',$$

$$= 2mm'x';$$

$$\therefore\ m'yy' \pm mzz' - 2mm'(x+x') = 0,$$

is the equation sought.

CHAP. VII.

ON THE ELLIPTIC PARABOLOID.

386. To find the nature of the section made by a plane parallel to any one of the co-ordinate planes.

If the intersecting plane be parallel to the plane of xy, its equation will be $z=\gamma$; whence, substituting this value of z in the equation

$$m'y^2+mz^2=4mm'x,$$

we have

$$m'y^2+m\gamma^2=4mm'x,$$

$$\text{or } m'y^2=m(4m'x-\gamma^2).$$

In like manner, if $y=\beta$, and $x=a$, the sections parallel to the planes of xz, and yz, are

$$mz^2=m'(4mx-\beta^2),$$

$$\text{and } m'y^2+mz^2=4mm'a.$$

It thence appears, that the sections parallel to the planes of xy and xz are *parabolas*, and that the sections parallel to yz are *ellipses*.

When $a=0$, the ellipse becomes a point, whence the surface touches the plane of yz at the origin.

When a is negative, the ellipse becomes imaginary; the surface, therefore, is wholly situated on one side of the plane yz.

387. Cor. The *principal sections* are

$$y^2 = 4mx \ldots\ldots\ldots (1),$$

$$z^2 = 4m'x \ldots\ldots\ldots (2),$$

$$m'y^2 + mx^2 = 0 \ldots\ldots (3).$$

The first two are *parabolas* whose common vertex is the origin, and whose common axis coincides with AX: the third is a point.

388. In the general equation

$$m'y^2 + mz^2 = 4mm'x.$$

Let $m = m'$; then

$$y^2 + z^2 = 4mx,$$

which (350) is the equation to the *paraboloid of revolution.*

This is the only *variety* of the elliptic paraboloid.

389. Prop. 2. If the parabola ($z^2 = 4m'x$) move parallel to itself, so that its vertex may always remain upon the parabola ($y^2 = 4mx$), it will generate the elliptic paraboloid.

The same notation being used as in Art. 252, the equation to the *generating* parabola is

$$z^2 = 4m'x \ldots\ldots\ldots (1),$$

and that to the directrix,

$$Y^2 = 4mX \ldots\ldots\ldots (2).$$

Since the generating parabola moves parallel to itself, its equations in any position will be

$$y = \beta, \quad z^2 = 4m'x + a.$$

But, since by hypothesis it passes through a point,

$$(x = X, \quad y = Y, \quad z = 0)$$

of the directrix, we have

$$X = -\frac{\alpha}{4m'} = \frac{4m'x - z^2}{4m'} = x - \frac{z^2}{4m'},$$

$$Y = \quad \beta \quad = y.$$

These values being substituted in (2), there results

$$m'y^2 + mz^2 = 4mm'x,$$

which is the equation to the *elliptic paraboloid*. Whence t truth of the proposition.

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CHAP. VIII.

ON THE HYPERBOLIC PARABOLOID.

390. **Prop. 1.** To find the nature of the section made by a plane parallel to any one of the co-ordinate planes.

Substituting for z, y, x, in the equation

$$m'y^2 - mz^2 = 4mm'x,$$

the quantities γ, β, α, we have

$$m'y^2 = m(4m'x + \gamma^2),$$

$$mz^2 = m'(\beta^2 - 4mx),$$

$$m'y'^2 - m^2 z = 4mm'x.$$

Whence it appears, that the sections made by planes parallel to the planes of xy and xz, are parabolas, whose common axis is that of x; and that the sections parallel to the plane of yz, are hyperbolas. If α be negative, the sections are still hyperbolas, whose equations are

$$mz^2 - m'y^2 = 4mm'\alpha.$$

If $\alpha = 0$, the hyperbola becomes two straight lines that intersect each other at the origin.

391. **Cor. 1.** *The principal sections* are

$$y^2 = 4mx \ldots\ldots\ldots (1),$$

$$z^2 = -4m'x \ldots\ldots\ldots (2),$$

$$m'y^2 - mz^2 = 0 \ldots\ldots\ldots (3).$$

The first two are parabolas, whose common vertex is the origin, and whose axes are AX, AX'. The third represents two straight lines which pass through the origin.

392. Cor. 2. To the *left* of A, take $AM'=x'$, then equation (2) becomes

$$z^2 = 4m'.AM' \ldots\ldots\ldots (1').$$

At this distance from the origin, let a plane be drawn parallel to the plane of zy, then the section will be an hyperbola, whose equation is

$$mz^2 - m'y^2 = 4mm'.AM'.$$

$$\text{Let } y=0, \text{ then } z^2=4m'.AM',$$

which values of z are those of the semi-axis major; but these are precisely the values of z in the parabola (1′): hence it follows that *the vertices of the hyperbola*

$$mz^2 - my'^2 = 4mm'x'$$

will be upon the parabola

$$z^2 = -4m'x'.$$

393. Cor. 3. No section of the hyperbolic paraboloid, made by a plane, can be an *ellipse*.

For if x be eliminated between the equation to any plane, and the equation

$$m'y^2 - mz^2 = 4mm'x,$$

the result will have no term which involves yz, wherefore (92) the curve cannot be an ellipse.

Hence, this species of paraboloid can never become a surface of revolution, since, in such case, the section by a plane perpendicular to the axis of revolution would be a circle, that is, a species of the *ellipse*; which is impossible.

The hyperbolic paraboloid presents no *variety*.

394. **Prop. 2.** To find the co-ordinates of the points in which the line

$$z = ax, \quad y = bx,$$

intersects the surface.

Substituting these values of y and z in the equation

$$m'y^2 - mz^2 = 4mm'x,$$

we have $(m'b^2 - ma^2)\,x^2 = 4mm'x,$

whence one value of x, and, therefore, of y and of $z = 0$.

$$\text{Also } x = \frac{4mm'}{m'b^2 - ma^2},$$

and y and z are known in like terms.

Consequently, the proposed line will meet the surface so long as the denominator $m'b^2 - ma^2$ is finite. When this quantity $= 0$, the line will be an asymptote; in which case

$$b = \pm a\sqrt{\frac{m}{m'}}.$$

The equation, therefore, to the rectilinear asymptote is

$$z = ax, \quad y = a\sqrt{\frac{m}{m'}}\,x.$$

Eliminating a, we obtain the equation

$$m'y^2 - mz^2 = 0,$$

$$\text{whence, } y = \pm z\sqrt{\frac{m}{m'}},$$

the equation to *two planes;* which, since they contain all the rectilinear asymptotes, will, themselves, be asymptotes to the surface.

395. Prop. 3. If the parabola $(z^2=-4m'x)$ move parallel to itself, so that its vertex describe the parabola $(y^2=4m'x)$, it will generate the hyperbolic paraboloid.

The equation to the generating parabola is $z^2=-4m'x$...(1),

and that to the directrix $Y^2=\ \ 4mX$...(2).

In any position the equations to the generating curve will be

$$y=\beta,\quad z^2=-4m'x+\alpha\ldots\ldots(3).$$

And, since it must pass through a point

$$(x=X,\quad y=Y,\quad z=0,)$$

of the directrix, we have in equation (3),

$$X=x=\frac{\alpha}{4m'}=\therefore\frac{x^2+4m'x}{4m'},$$

$$Y=y;$$

therefore, equation (2) becomes, by substitution,

$$y^2=\frac{m}{m'}(z^2+4m'x);$$

$$\therefore\ m'y^2-mz^2=4mm'x,$$

which is the equation to the hyperbolic paraboloid.

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CHAP. IX.

ON THE SECTIONS OF SURFACES OF THE SECOND ORDER MADE BY A PLANE.

396. PROP. 1. To find the nature of the curve formed by the intersection of a surface and a plane.

I. Let the surface be supposed to have a centre.

Then, the general equation is, (353)

$$Ax^2+By^2+Cz^2+D=0.$$

In order to determine the section made by a plane, we must substitute for x, y, z, the following values (321):

$$x=x'\cos\phi+y'\cos\theta\cos\phi,$$

$$y=x'\sin\phi-y'\cos\theta\cos\phi,$$

$$z=y'\sin\theta.$$

The result, developed and arranged, will be

$$(A\cos^2\phi+B\sin^2\phi)\,x'^2+\{C\sin^2\theta+\cos^2\theta\,(A\sin^2\phi+B\cos^2\phi)\}y'^2$$
$$+2\,(A-B)\sin\phi\cos\phi\cos\theta\,.\,x'y'+D=0\ldots\ldots(e).$$

Wherefore, the curve of intersection is *a line of the second order*.

The *species* of the curve will depend, (Art. 90 and 201), on the sign of the quantity

$$A-B^2)\sin^2\phi\cos^2\phi\cos^2\theta-4\,(A\cos^2\phi+B\sin^2\phi)\,\{C\sin^2\theta+\cos^2\theta(A\sin^2\phi+B\cos^2\phi)\},$$

or, when the expression is reduced, of the quantity

$$-AB\cos^2\theta-BC\sin^2\theta\sin^2\phi-AC\sin^2\theta\cos^2\phi\ \ldots\ldots\ (1).$$

Let us now suppose the surface to be

(1) An ellipsoid.

Then, the coefficients A, B, C being all positive, the above quantity is essentially negative. Consequently, the section of an ellipsoid, by a plane, must be an *ellipse*, or *one of its varieties.*

(2) An hyperboloid of *one*, or of *two* sheets.

Then, since any two of the coefficients A, B, C may be positive or negative, and the remaining one negative or positive, the above quantity (1), may be positive, or negative, or $=0$. Consequently, the section of either species of hyperboloid by a plane, may be an *ellipse*, *hyperbola*, or *parabola.*

We have already arrived at the same conclusion, (Art. 344, and 345), in the case of the *cone*, which, in fact, is only a variety of either kind of hyperboloid.

II. Let the surface be supposed not to have a centre.

Then, the general equation is of the form

$$My^2 + M'z^2 + Px = 0.$$

Following the same steps as in the first part of the proposition, we obtain, for the equation to the curve of intersection,

$$(M\cos^2\theta\cos^2\phi + M'\sin^2\theta)y^2 + M'\sin^2\phi \,.\, x^2 - 2M\sin\phi\cos\phi\cos\theta . xy$$

$$+ P\cos\phi\cos\theta \,.\, y + P\cos\phi \,.\, x = 0 \;\ldots\ldots\; (e').$$

The curve is, therefore, a line of the second order, whose species depends on the sign of the quantity

$$-MM'\sin^2\phi\sin^2\theta.$$

(1) Let the surface be the *elliptic* paraboloid.

Then because M, M' have the same sign, the above quantity is essentially negative. Hence, the section is an *ellipse.* The quantity $-MM' \sin^2\phi \sin^2\theta$ will $=0$, if either θ or $\phi=0$. The sections, therefore, in either case, will be *parabolas*, (208).

Consequently, the sections of the elliptic paraboloid, by a plane, may be an *ellipse* or *parabola*, or a *variety of either curve*.

(2) Let the surface be the *hyperbolic* paraboloid.

Then, M, M' having different signs, the quantity

$$-MM' \cos^2\phi \sin^2\theta$$

is necessarily positive. Whence, the section is an *hyperbola*. If, however, ϕ or $\theta=0$, the section will, in either case, be a *parabola*. Consequently, the section of the *hyperbolic* paraboloid, may be an *hyperbola* or a *parabola*, or *a variety of either curve*.

397. COR. It appears, from the second case of the Proposition, that the section of the *elliptic* paraboloid can never be an *hyperbola*, nor the section of the *hyperbolic* paraboloid, an *ellipse*.

398. PROP. 2. To investigate the conditions under which the section of a surface of the second order, may be a *circle*.

I. Let the surface have a centre.

Then, in order that the section may be a circle, the coefficients of y^2 and x^2, in the equation to the section, must be equal, and the term involving xy, must vanish, (204).

Whence, referring back to equation (e), Art. 396,

$$A\cos^2\phi+B\sin^2\phi=C\sin^2\theta+\cos^2\theta\,(A\sin^2\phi+B\cos^2\phi) \ldots (m)$$

$$\text{and } 2(A-B)\sin\phi\cos\phi\cos\theta=0 \ldots\ldots\ldots\ldots\ldots\ldots (n)$$

Now the latter of these equations will be satisfied, if

(1) $\sin\phi = 0$, or $\angle\ t, x = 0$, (321, *Note*).

(2) $\cos\phi = 0$, or $t, x = \frac{\pi}{2}$.

(3) $\cos\theta = 0$, or $xy, x'y' = \frac{\pi}{2}$.

(1) Suppose $\phi = 0$, or *that the plane is perpendicular to that of yz.*

Then equation (m) becomes

$$C \sin^2\theta + B \cos^2\theta = A,$$

or, dividing each term by $\cos^2\theta$,

$$C \tan^2\theta + B = \frac{A}{\cos^2\theta} = A\,(1 + \tan^2\theta)\,;$$

$$\text{therefore } \tan\theta = \pm\sqrt{\frac{B-A}{A-C}} \;\ldots\ldots(1).$$

(2). Let $\phi = \frac{\pi}{2}$, or *the plane be perpendicular to that of xz.*

Then, equation (m) becomes

$$C \sin^2\theta + A \cos^2\theta = B.$$

$$\text{Whence, } \tan\theta = \pm\sqrt{\frac{A-B}{B-C}} \;\ldots\ldots(2).$$

(3) Let $\theta = \frac{\pi}{2}$, or *the plane be perpendicular to that of xy.*

Then, as in the first two cases, there results

$$\tan\phi = \pm\sqrt{\frac{A-C}{C-B}} \;\ldots\ldots(3).$$

Now, the three quantities $\frac{B-A}{A-C}$, $\frac{A-B}{B-C}$, $\frac{A-C}{C-B}$ are so related to one another, that only *one* of them can be positive. For the product of all three being $\left(\frac{A-B}{B-C}\right)^2$, a quantity necessarily positive, it follows, that one, *at least*, of the factors, must also be positive. Let the first, $\frac{B-A}{A-C}$, be supposed positive; then, $B-A$ and $A-C$, and, therefore, their sum $B-C$, must have the *same sign;* consequently, $B-C$ and $A-B$; $C-B$ and $A-C$ must have *different signs;* that is, the second and third must be negative.

Similarly it may be shewn, that if the second or third be supposed positive, the remaining two will, in each case, be negative.

Hence, of the three quanties (1), (2) and (3), one only is *real,* and this, having a double sign, furnishes *two* values of the corresponding tangent.

II. Let the surface be supposed not to have a centre.

Then, as before, the section will be a circle, when, in equation (e'),

$$M\cos^2\theta\cos^2\phi + M'\sin^2\theta = M'\sin^2\phi \,\ldots\ldots (m'),$$

$$\text{and}\quad 2M\sin\phi\cos\phi\cos\theta = 0 \,\ldots\ldots\ldots\ldots (n').$$

Equation (n') will be satisfied, by supposing

$$\phi = 0,\quad \text{or}\ \phi = \frac{\pi}{2},\quad \text{or}\ \theta = \frac{\pi}{2}.$$

But, if ϕ be supposed $=0$, the coefficient of x^2, in equation (e'), would $=0$, which, when the section is a circle, cannot be the case. Whence, we can only admit the two last hypotheses.

(1) Let $\phi = \frac{\pi}{2}$, or the *plane be perpendicular to that of xz.*

Then, equation (m') becomes

$$M' \sin^2 \theta = M';$$

$$\text{therefore } \sin \theta = \pm \sqrt{\frac{M'}{M}} \ldots\ldots (1).$$

(2) Let $\theta = \frac{\pi}{2}$, or *the plane be perpendicular to that of xy.*

Then, we have, in like manner,

$$\sin \phi = \pm \sqrt{\frac{M}{M'}} \ldots\ldots (2).$$

In the *elliptic* paraboloid, *one* of these quantities will be *real,* and the other *imaginary,* according as M is greater or less than M'; the real quantity will furnish, as before, two values of the corresponding sine.

In the case of the *hyperbolic* paraboloid, M and M' having different signs, the values of $\sin\theta$, and $\sin\phi$, will be imaginary; whence, no plane can be drawn, which shall intersect this surface in a circle. See Art. 393.

We conclude, therefore, that *in all surfaces of the second order,* except the hyperbolic paraboloid, *two planes may be drawn, such, that their intersection with the surface shall be a circle.*

The *sub-contrary section* of the cone (346), is to be regarded as a particular case of this Proposition.

399. Cor. 1. In finding the nature of the section, Art. 396, suppose that the *origin,* as well as the direction of the axes is changed, and let α, β, γ be the co-ordinates of the new origin.

Then, since the equations of condition (m), (n), and (m'), (n'), which determine the section to be a circle, are independent of the quantities α, β, γ, the Proposition just proved will hold true, whatever be the point through which the intersecting plane is drawn. Whence it follows, that the pairs of planes which intersect the surface in circles, will be infinite in number, and parallel to one another.

400. Cor. 2. If we replace the indeterminate quantities A, B, C by their values $-\frac{D}{a^2}, -\frac{D}{b^2}, -\frac{D}{c^2}$ (354), and then give to a^2, b^2, c^2 their proper signs, according as the surface belongs to one or other of the three species (362), we shall have no difficulty in proving, that the sections are circles in each of the following cases:

(1) In the *ellipsoid.*

When the intersecting plane passes through the mean axis $2b$, and is inclined to the plane of xy, at angles

$$=\tan^{-1} \pm \frac{c}{a} \sqrt{\frac{a^2-b^2}{b^2-c^2}}.$$

(2) In the *hyperboloid of one sheet.*

When the intersecting plane passes through the greatest axis $2a$, and forms with the plane of xy angles

$$=\tan^{-1} \pm \frac{c}{b} \sqrt{\frac{a^2-b^2}{a^2+c^2}}.$$

(3) In the *hyperboloid of two sheets.*

When the intersecting plane is parallel to that which passes through the mean axis $2b$, and forms with the plane of xy angles

$$=\tan^{-1} \pm \frac{c}{a} \sqrt{\frac{a^2+b^2}{b^2-c^2}}.$$

401. Prop. 3. Surfaces of the second order, with the exception of the hyperbolic paraboloid, may be generated in two

different ways, by the motion of a circle parallel to itself, and of variable radius.

Having already shewn, that an infinite number of parallel planes may be drawn, intersecting the surfaces in circles, it only remains to prove, that the centres of these circles will be situated on the same straight line.

For this purpose, let the origin be transferred to the centre of any one of the circular sections: the terms involving x' and y', in equations (e), and (e'), Art. 396, will, therefore, vanish (91), and we shall have

$$C\gamma \sin\theta - B\beta \cos\theta \cos\phi + A\alpha \cos\theta \sin\phi = 0,$$

$$B\beta \sin\phi + A\alpha \cos\phi \ldots\ldots = 0.$$

Each of these equations being *linear*, it follows, that the locus of the centre is a straight line. Whence the truth of the Proposition.

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CHAP. X.

ON THE CONDITIONS WHICH DETERMINE A SURFACE OF THE SECOND ORDER TO BE A SURFACE OF REVOLUTION.

402. Prop. 1. To find what must be the relation among the coefficients of the general equation, in order that it may represent a *surface of revolution.*

The principle of the transformation of co-ordinates, leads, in a very simple manner, to the solution of the proposed question.

The surface being referred to rectangular axes, originating at the centre, its equation will be

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'xz + 2C'xy = D \dots (1).$$

Let x', y', z' be the co-ordinates of another rectangular system, which has the same origin as the first.

Then, supposing the surface to be one of revolution about the axis z', its equation will be of the form

$$x'^2 + y'^2 = F(z'),$$

or, since the centre is the origin, (350)

$$x'^2 + y'^2 = az'^2 + b \dots (2).$$

a and b being constant quantities.

If we now transform the co-ordinates from this, to the primitive system, the transformed equation will be identical with equation (1).

Replacing, therefore, x', y' and z', by their values in (316), and denoting cos x', x; cos x', y; cos x', z, &c. by m, n, p, &c., equation (2) becomes

$$(mx+ny+pz)^2+(m'x+n'y+p'z)^2=a(m''x+n''y+p''z)^2+b;$$

whence, developing and arranging the expression, we have

$$(m^2+m'^2-am''^2)x^2+(n^2+n'^2-an''^2)y^2+(p^2+p'^2-ap''^2)z^2$$
$$+2(np+n'p'-an''p'')yz+2xz(mp+m'p'-am''p'')$$
$$+2(mn+m'n'-am''n'')xy=b.$$

But, (244) $m^2+m'^2+m''^2=1$;

$$\therefore\ m^2+m'^2-am''^2=1-(1+a)m''^2$$
$$=1-\alpha^2,$$

if α^2 be taken $=(1+a)m''^2$.

In like manner, by assuming

$$(1+a)n''^2=\beta^2,\ \text{and}\ (1+a)p''^2=\gamma^2$$

we may prove, that

$$n^2+n'^2-an''^2=1-\beta^2,$$

$$\text{and}\ p^2+p'^2-ap''^2=1-\gamma^2.$$

Again, $mn+m'n'+m''n''=0$, (247);

$$\therefore\ mn+m'n'-am''n''=-(1+a)m''n''$$
$$=-\alpha\beta,$$

and, similarly, for the other terms; wherefore the transformed equation becomes

$$(1-\alpha^2)x^2+(1-\beta^2)y^2+(1-\gamma^2)z^2-2\beta\gamma.yz-2\alpha\gamma.xz-2\alpha\beta.xy=b.(3).$$

Hence, equating the corresponding terms of (1) and (3), we have

$$\frac{1-\alpha^2}{b}=\frac{A}{D};\qquad \text{let } \frac{b}{D}=\delta,$$

Then, $1-\alpha^2=\ \delta A$; and, similarly, $1-\beta^2=\ \delta B,\ 1-\gamma^2=\ \delta C.$ (4),

Also $\beta\gamma=-\delta A'$ .... $\alpha\gamma=-\delta B'$; $\alpha\beta=-\delta C'$ (5).

Multiplying together each two of equations (5), and dividing the product by the third, we have

$$-\frac{A'C'}{B'}\delta=\beta^2,$$

$$-\frac{A'B'}{C'}\delta=\gamma^2,$$

$$-\frac{B'C'}{A'}\delta=\alpha^2;$$

whence, substituting for α^2, β^2, γ^2 their values in (4), and eliminating δ, there results

$$\frac{AA'-B'C'}{A'}=\frac{BB'-C'A'}{B'}=\frac{CC'-A'B'}{C'};$$

whence, $A'C'(C-A)-B'(C'^2-A'^2)=0,$

and $A'C'(C-B)-A'(C'^2-B'^2)=0,$

which are the equations of condition required.

From this result it appears, that the generality of the conclusion is not restricted by our having assumed the centre as the origin.

403. Cor. 1. If any one of the terms involving the product of the variables be wanting, the surface cannot be one of revolution.

Let A', for example, $=0$.

Then, the equations of condition are reduced to this,

$$B'C'^2=0;$$

whence, $B'=0$, or $C'=0$; that is, the general equation will want *two* terms involving the products; which is contrary to the hypothesis.

404. Cor. 2. Let *two* of the coefficients, A' and B', for example, $=0$.

Then, from the first of the equations of condition, we have

$$\frac{B'}{A'}=\frac{C-A}{C'}.$$

And from the second,

$$\frac{B'}{A'}=\frac{C'}{C-B};$$

therefore, these equations become

$$C'^2=(C-A)(C-B).$$

Cor. 3. If, moreover, $C'=0$.

Then, from the equation just deduced, there must result either

$$C=A, \text{ or } C=B;$$

that is, when all the three terms involving the products are wanting, the surface cannot be one of revolution, unless two of the coefficients A, B, C be equal, and have the same sign.

405. Prop. 2. To find the equations to the axis of revolution.

The equation to z' being $x'=0$, $y'=0$, we have

$$\left.\begin{array}{l} mx+ny+pz=0 \\ m'x+n'y+p'z=0 \end{array}\right\} \quad \ldots\ldots\ldots \quad (1).$$

Multiplying the first by m, the second by m', and taking the sum of the products, we have

$$(m^2+m'^2)\,x+(mn+m'n')\,y+(mp+m'p')+z=0;$$

therefore, adding $m''^2x+m''n''y+m''p''z$ to each side,

$$(m^2+m'^2+m''^2)\,x+(mn+m'n'+m''n'')\,y+(mp+m'p'+m''p'')\,z$$
$$=m''\,(m''x+n''y+p''z);$$

hence, (244 and 247)

$$x=m''\,(m''x+n''y+p''z).$$

In like manner, multiplying, successively, the first of equations (1), by n, p, and the second by n', p', and taking the sum of the respective products, we have

$$y=n''\,(m''x+n''y+p''z),$$
$$z=p''\,(m''x+n''y+p''z).$$

Whence, eliminating the factor which is common to the last three equations, there results.

$$\frac{x}{m''}=\frac{y}{n''}=\frac{z}{p''},$$

or (402), since

$$m''=\frac{\alpha}{\sqrt{1+\alpha}},\quad n''=\frac{\beta}{\sqrt{1+\alpha}},\quad \text{and } p''=\frac{\gamma}{\sqrt{1+\alpha}},$$

$$\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}.$$

But, by the last Proposition,

$$-A'B'C'\delta = B'^2\beta^2 = C'^2\gamma^2 - A'^2\alpha;$$

$$\therefore\ A'\alpha = B'\beta = C'\gamma;$$

$$\therefore\ A'x = B'y = C'z;$$

$$\text{or } x = \frac{C'}{A'}z, \quad y = \frac{C'}{B'}z,$$

which are the equations required.

On the subject of this Chapter, the reader may consult *Annales de Mathématiques,* tom. VII. 174, and also *Correspondance Sur L'E'cole Polytechnique,* tom. II. 187.

ERRATA.

Page 22. last line, *for* $\frac{ch}{c+h} : \frac{bh'}{b+h'}$ *read* $\left(\frac{ch}{c+h}\right)^2 : \left(\frac{bh'}{b+h'}\right)^2$.
74. last line, *for* $(y-y')$ *read* $(y-y')^2$.
107. last line, *for* $a'^2b'^2$ *read* $\pm a'^2b'^2$.
116. 3 *add* fig. 50.
118. 15 *for* $a^2yy'+b^2xx'=a^2b^2$ *read* $a'^2yy'+b'^2xx'=a'^2b'^2$.
120. 19 *for* $\sqrt{\left\{a^2-\frac{a^2-b^2}{a^2}x^2\right\}}$ *read* $\frac{b}{a}\sqrt{\left\{a^2-\frac{a^2-b^2}{a^2}x^2\right\}}$.
147. 4 from bottom, *for* MA *read* $2\,MA$.
152. 2 *for* CX *read* CY.
156. 8 from bottom, *for* $-2ax$ *read* $+2ax$.
173. 3 from bottom, *for* fig. 63 *read* fig. 66.
176. 11 from bottom, *for* fig. 64 *read* fig. 67.
184. 4 from bottom, *for* fig. 66 *read* fig. 70.
212. 18 *erase* fig. 79.
213. 10 *for* $\cos\rho, r$ *read* $\cos\rho, x$,
249. 16 *for* $y'xt$ *read* $y'zt$.
267. 12 *for* $=c^2$ *read* $=0$,

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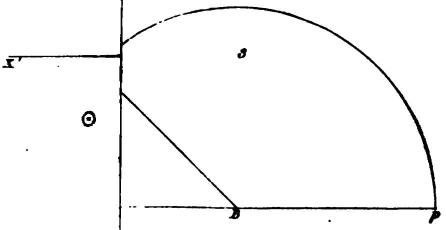

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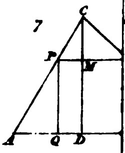

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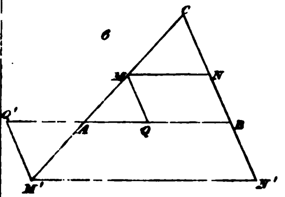

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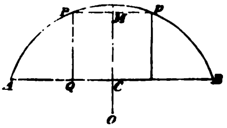

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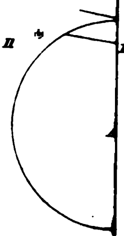

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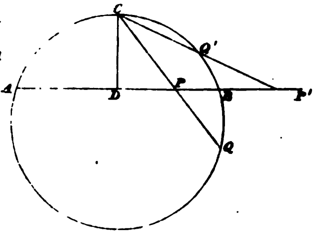

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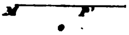

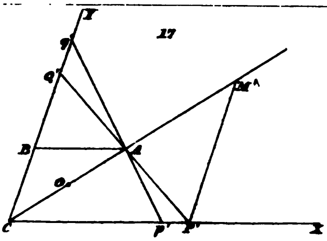

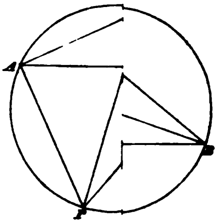

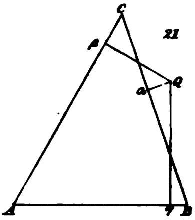

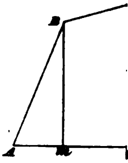

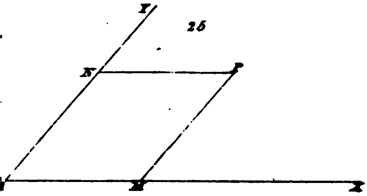

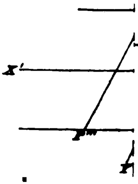

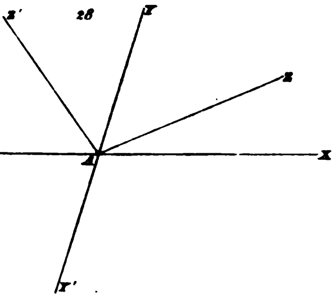

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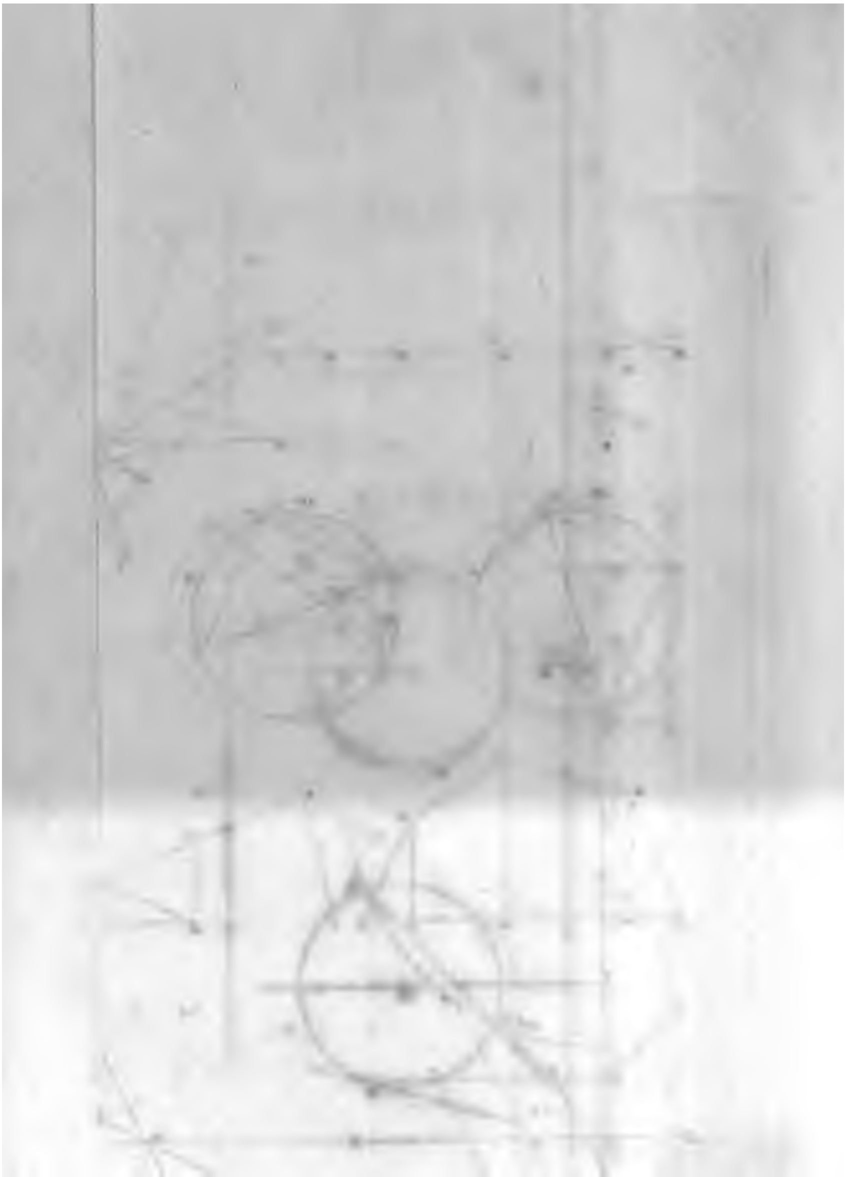

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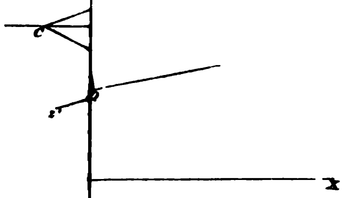

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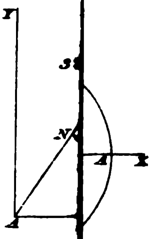

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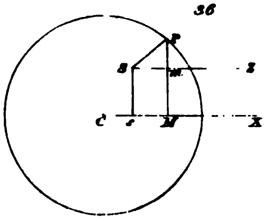

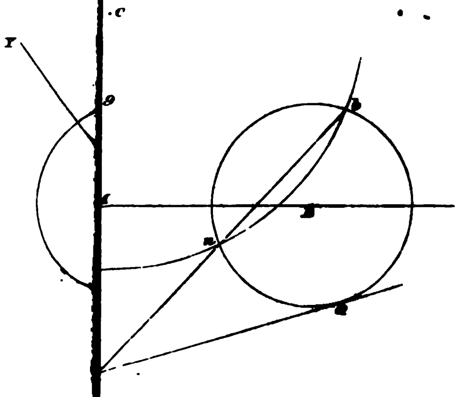

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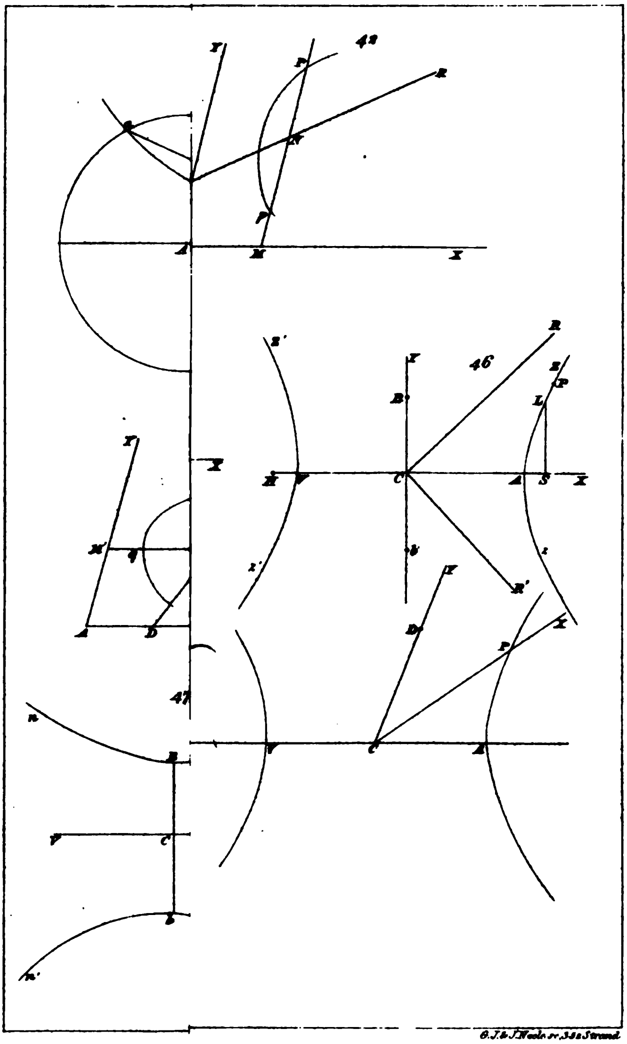

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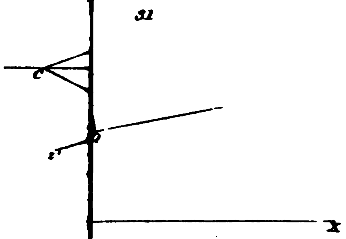

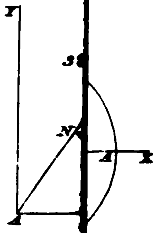

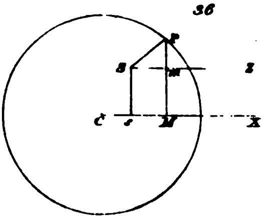

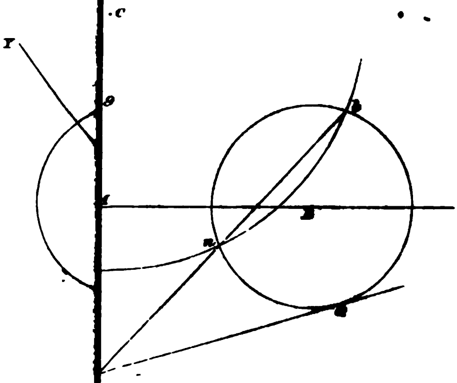

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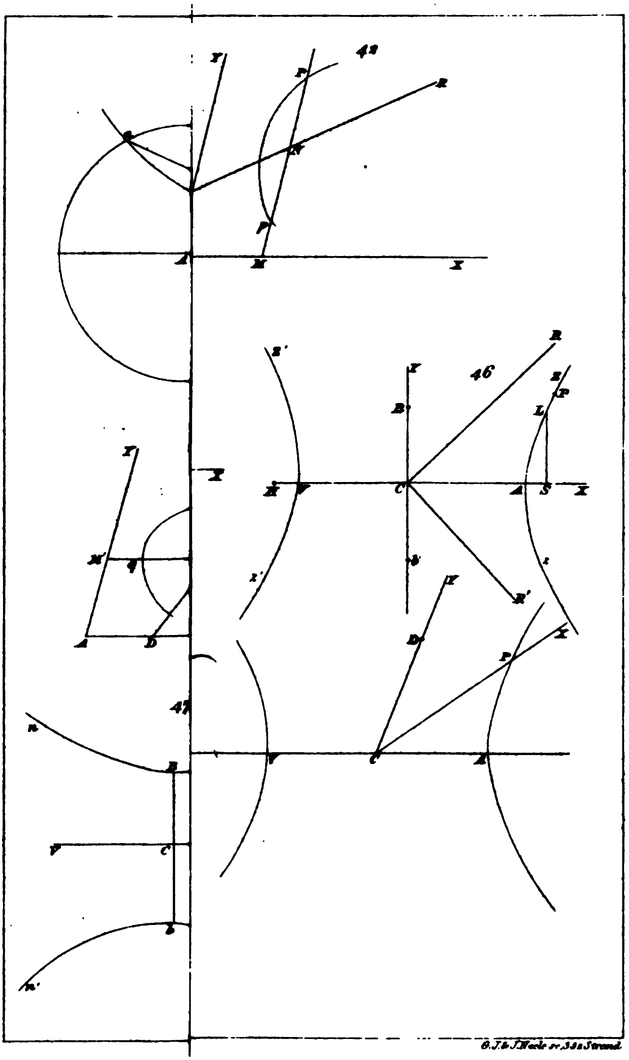

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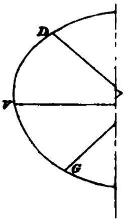

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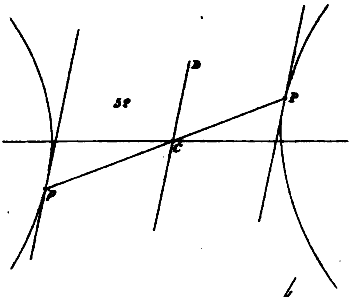

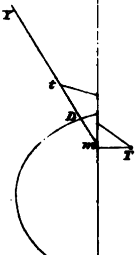

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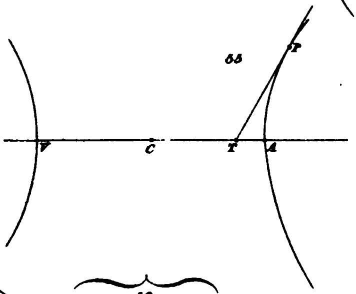

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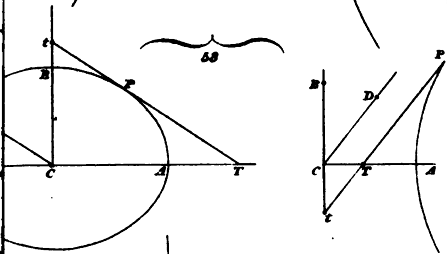

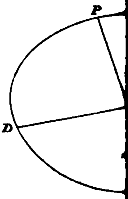

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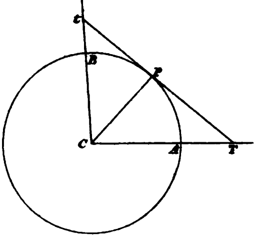

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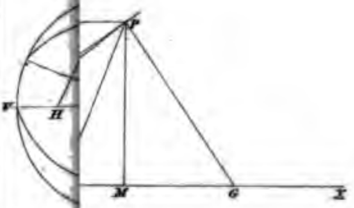

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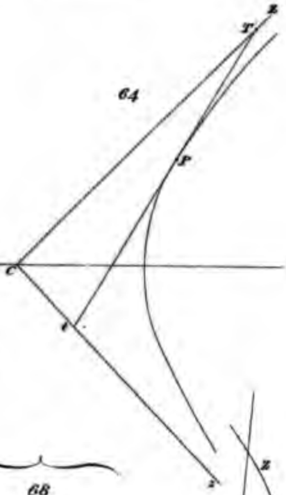

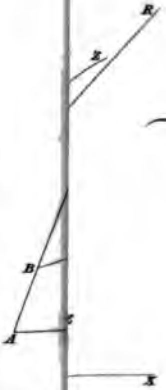

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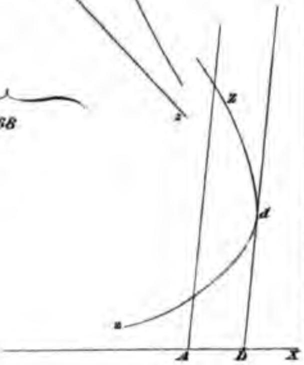

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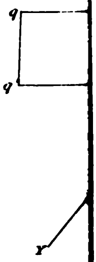

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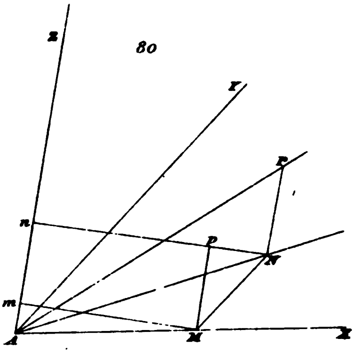

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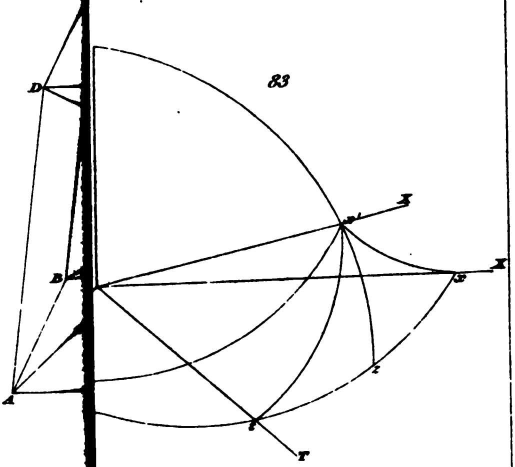

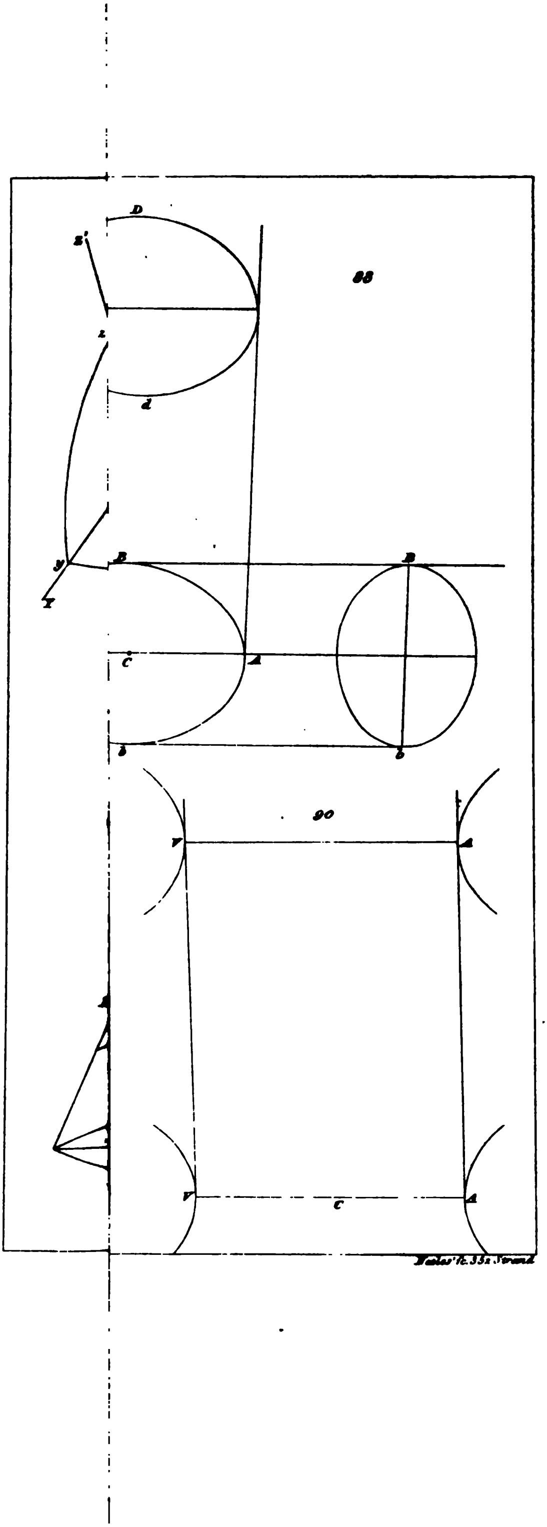

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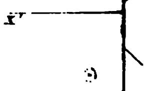

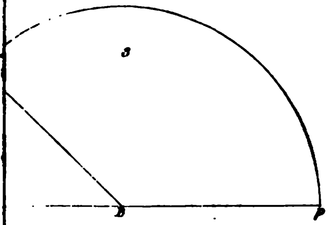

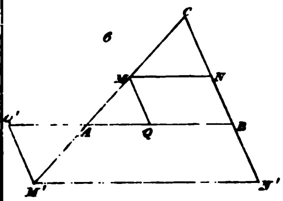

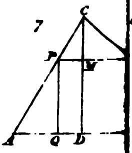

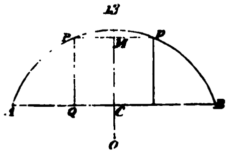

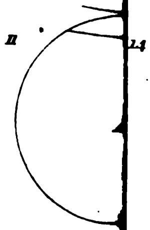

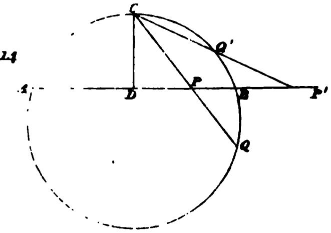

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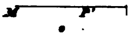

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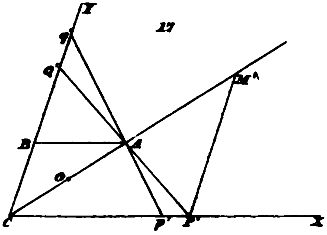

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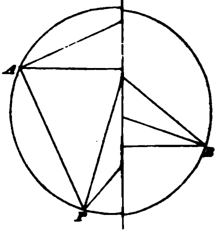

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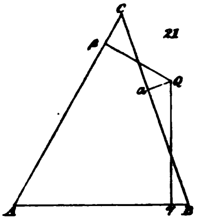

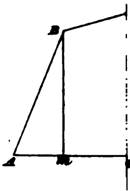

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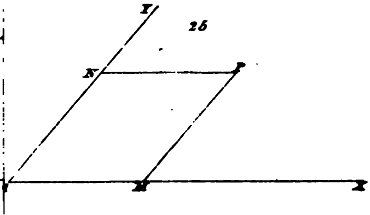

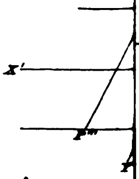

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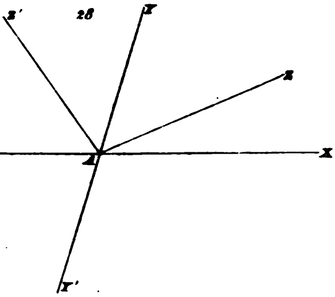

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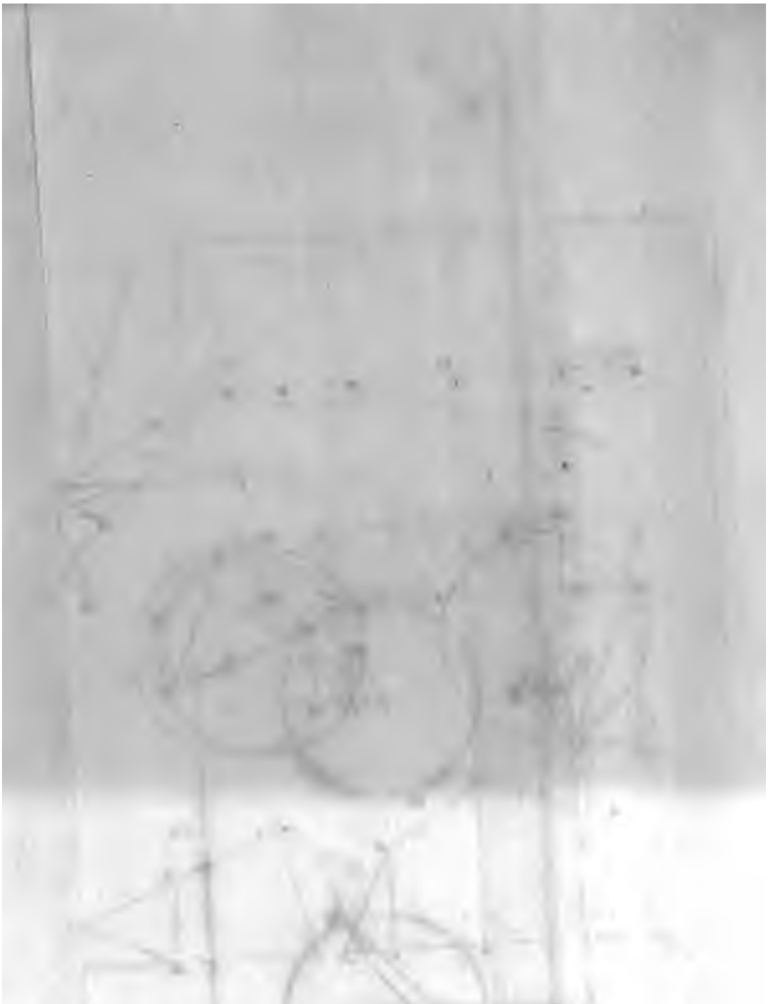

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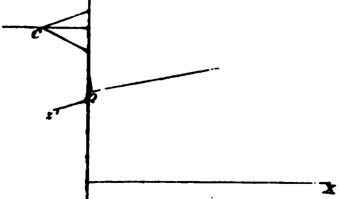

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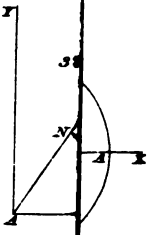

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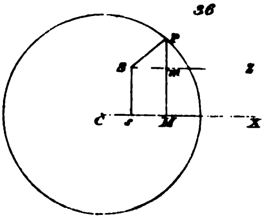

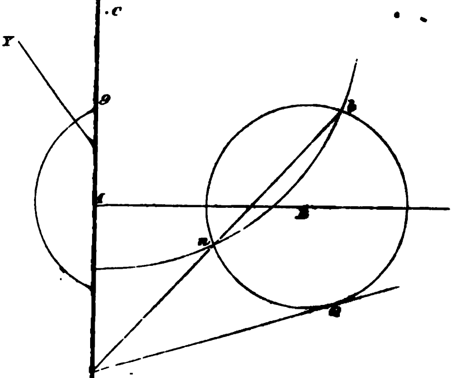

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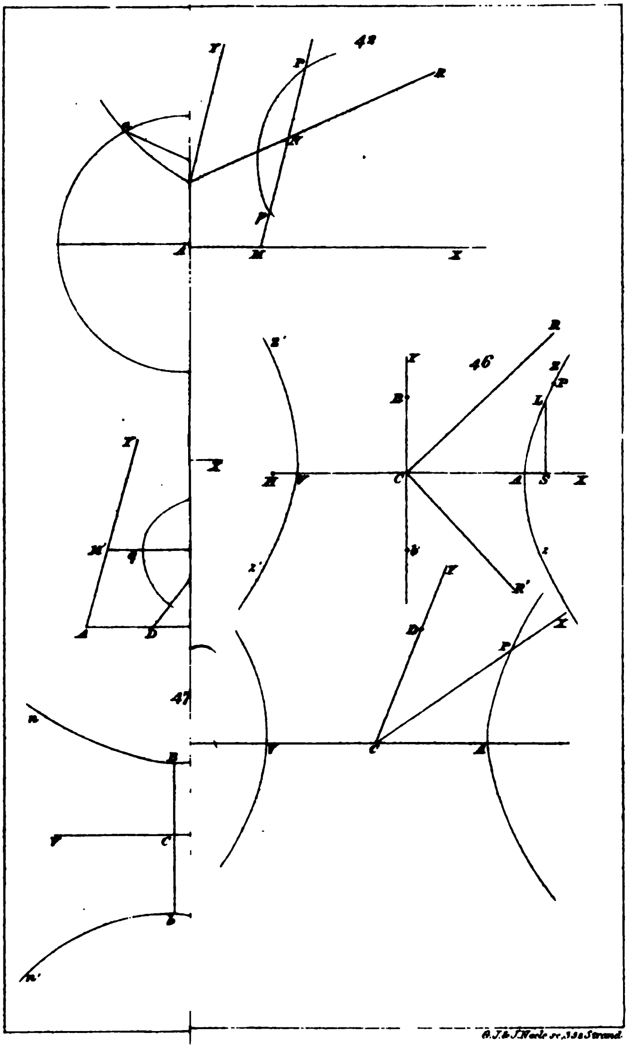

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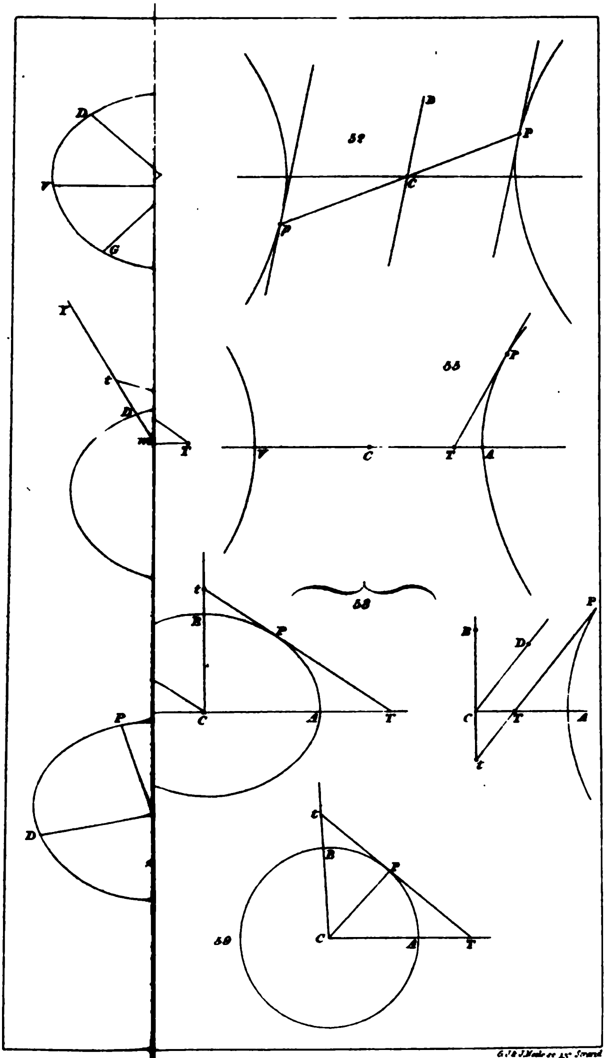

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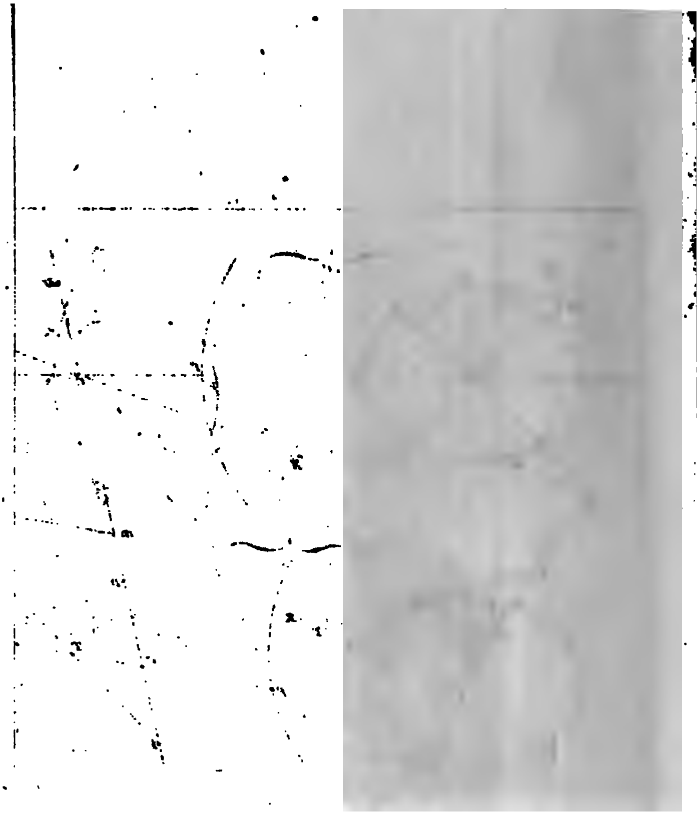

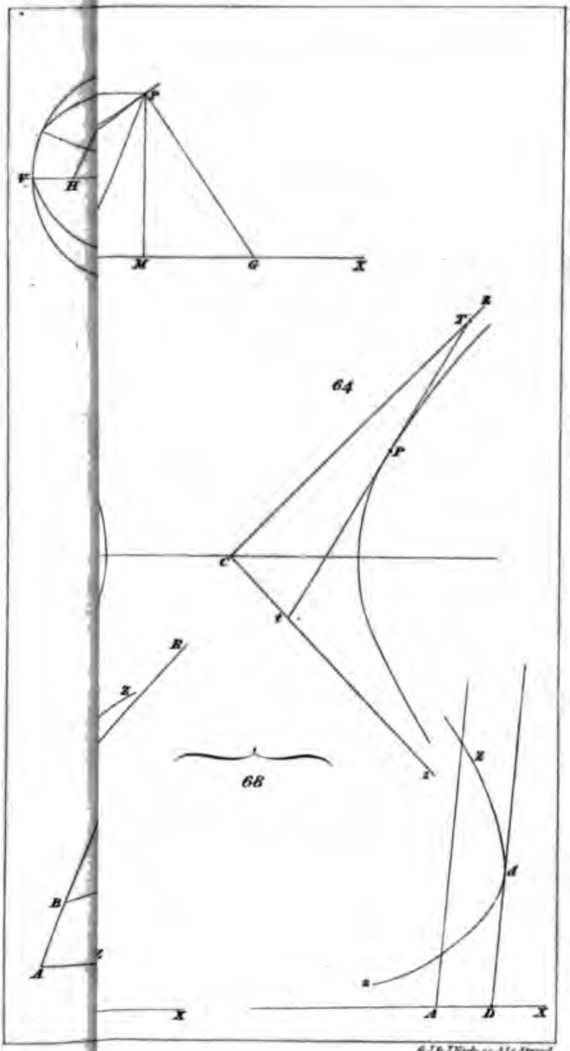

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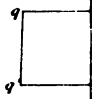
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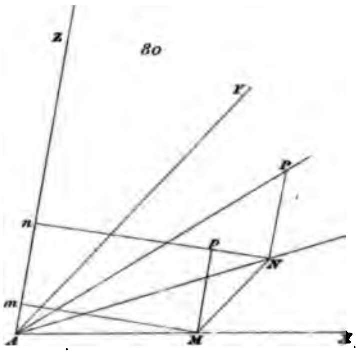
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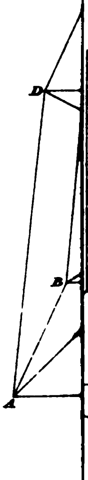
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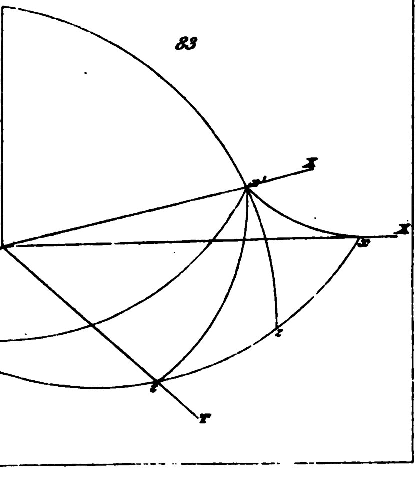
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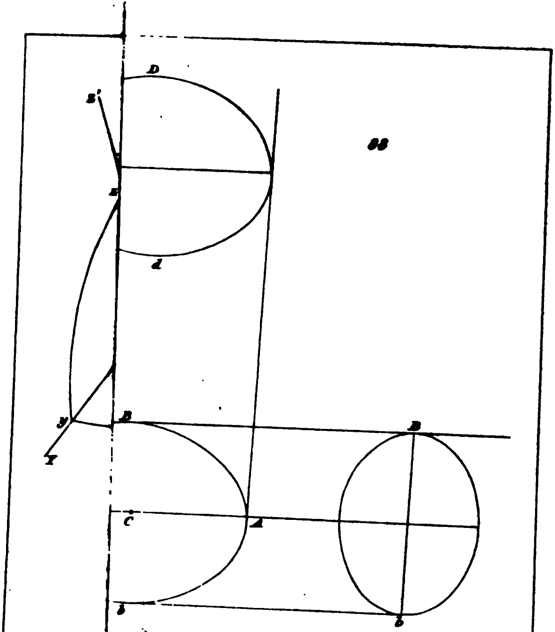
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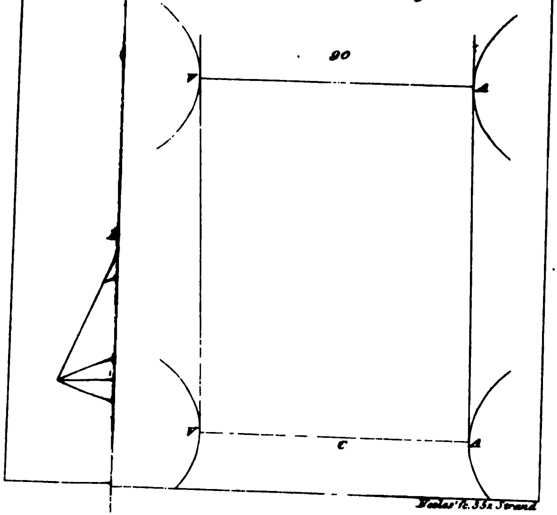
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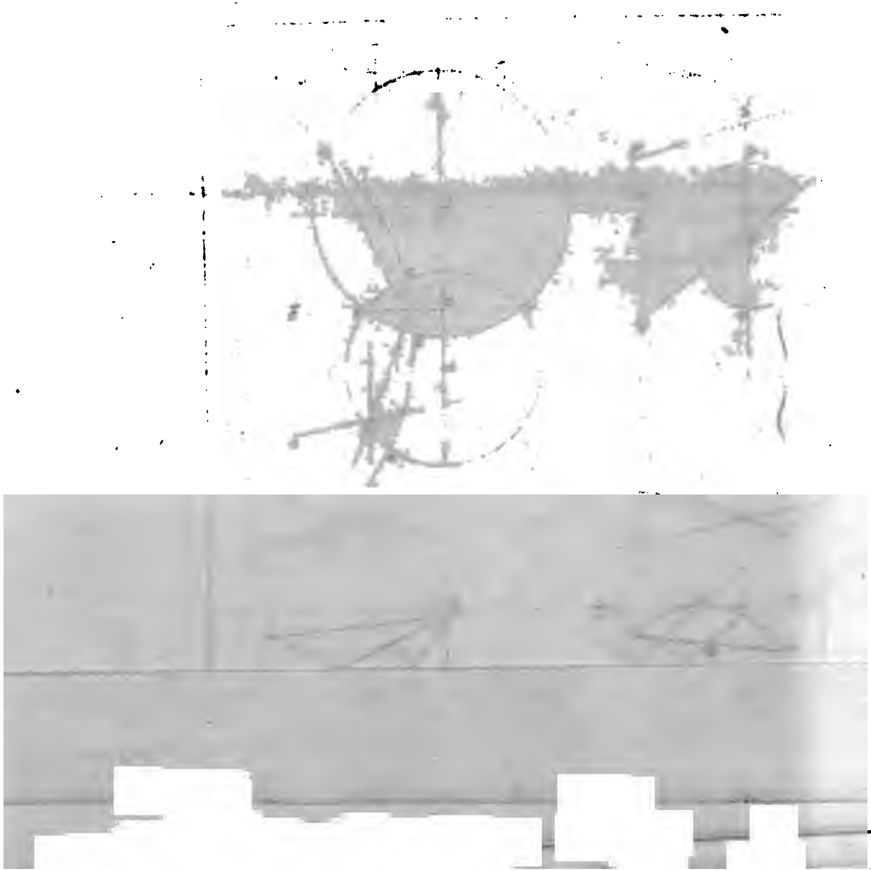

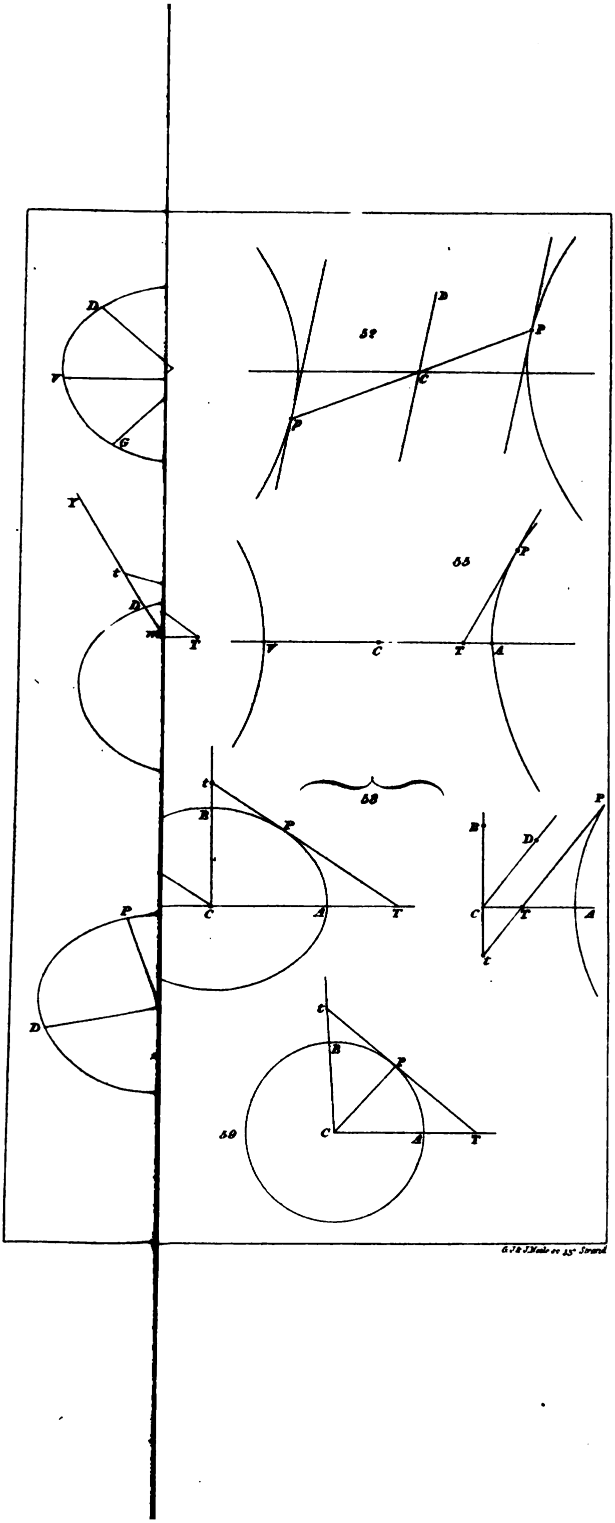

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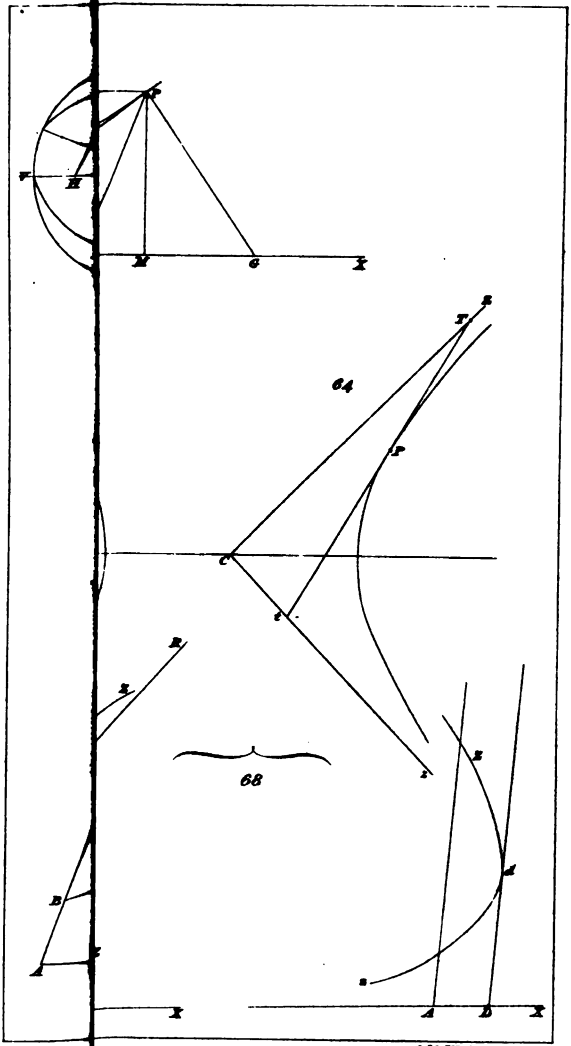

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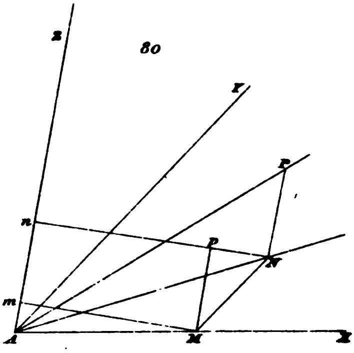

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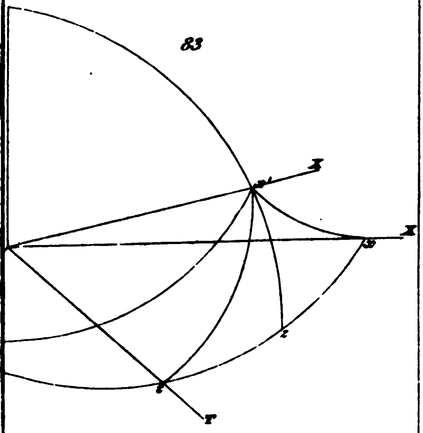

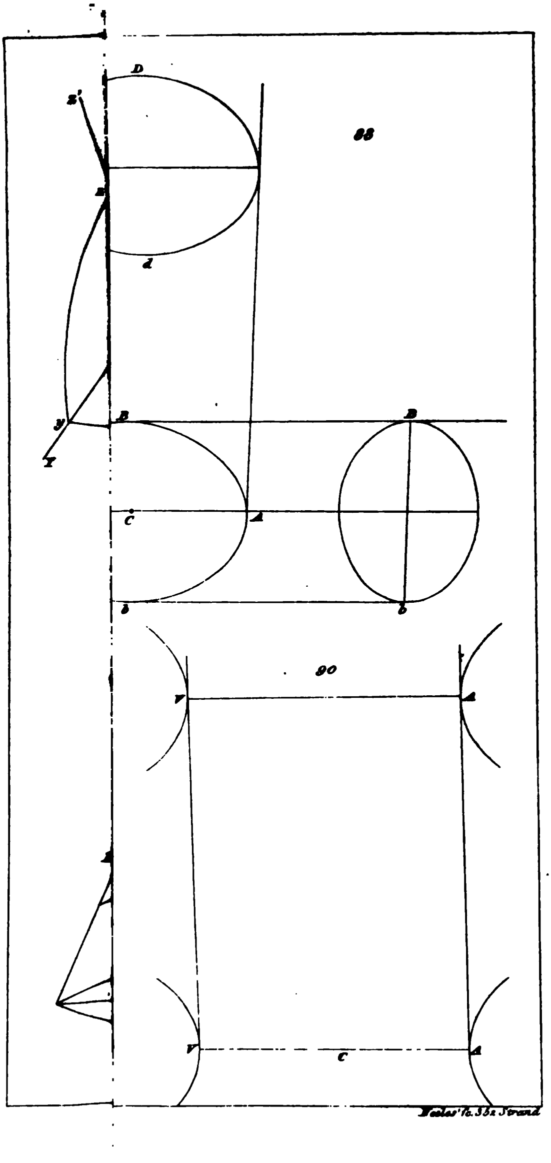

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